On Stern-Gerlach forces allowed by special relativity and the special case of the classical spinning particle of Derbenev-Kondratenko

K. Heinemann*

Deutsches Elektronen-Synchrotron DESY , Hamburg

February 2, 2008

Abstract

This work is devoted to an examination of Stern-Gerlach forces consistent with special relativity and is motivated by recent interest in the relativistic Stern-Gerlach force acting on polarized protons in high-energy particle accelerators. The equations for the orbital and spin motion of a classical charged particle with arbitrary intrinsic magnetic dipole moment in an external electromagnetic field are considered and by imposing the constraints of special relativity and restricting to first order in spin (= first order $\hbar$) a well-defined class of spin-orbit systems is obtained. All these systems can be treated on an equal footing including such prominent cases as those considered by Frenkel and by Good.

The Frenkel case is considered in great detail because I show that this system is identical with the one introduced by Derbenev and Kondratenko for studying spin motion in accelerators. In particular I prove that the spin-orbit system of Derbenev and Kondratenko is (nonmanifestly) Poincaré covariant and identify the transformation properties of this system under the Poincaré group. The Derbenev-Kondratenko Hamiltonian was originally proposed as a way to combine relativistic spin precession and the Lorentz force. The aforementioned findings now demonstrate that the Derbenev-Kondratenko Hamiltonian also provides a legitimate framework for handling the relativistic Stern-Gerlach force.

Numerical examples based on the Frenkel and Good cases for the HERA proton ring and electromagnetic traps are provided.

---

*heineman@mail.desy.de

Contents

Introduction

One of the most economical descriptions of the motion of polarized beams of spin $1/2$ particles in high-energy accelerators is provided by the semiclassical Hamiltonian given by Derbenev and Kondratenko [DK73], which in the language of the Dirac equation can be written in the form [1]:

$$H_{M,op} \equiv H_{M,op,orb} + H_{M,op,spin} \ , \tag{0.1}$$

with

$$H_{M,op,orb} = \beta \cdot \sqrt{\vec{\pi}^\dagger_{M,op} \cdot \vec{\pi}_{M,op} + m^2} + e \cdot \phi_{M,op} \ ,$$
$$H_{M,op,spin} = \vec{\sigma}^\dagger_{op} \cdot \vec{W}_{M,op} \ , \tag{0.2}$$

where

$$\vec{\pi}_{M,op} = \vec{p}_{M,op} - e \cdot \vec{A}_{M,op} \equiv \ canonical \ momentum \ vector \ ,$$
$$\vec{W}_{M,op} = -\frac{e}{2 \cdot m} \cdot \left([\frac{m}{J_{M,op}} + \frac{g-2}{2}] \cdot \beta \cdot \vec{B}_{M,op} \right.$$
$$-\frac{g-2}{2} \cdot \frac{1}{J_{M,op} \cdot (J_{M,op} + m)} \cdot \beta \cdot \vec{\pi}^\dagger_{M,op} \cdot \vec{B}_{M,op} \cdot \vec{\pi}_{M,op}$$
$$\left. -[\frac{g}{2 \cdot J_{M,op}} - \frac{1}{J_{M,op} + m}] \cdot (\vec{\pi}_{M,op} \wedge \vec{E}_{M,op}) \right) + \text{hermitian conjugate} \ , \tag{0.3}$$

with

$$J_{M,op} = \sqrt{\vec{\pi}^\dagger_{M,op} \cdot \vec{\pi}_{M,op} + m^2} \ . \tag{0.4}$$

Furthermore I define

$$\vec{v}_{M,op} \equiv \frac{i}{\hbar} \cdot (H_{M,op} \cdot \vec{r}_{M,op} - \vec{r}_{M,op} \cdot H_{M,op}) \ ,$$
$$\gamma_{M,op} \equiv (1 - \vec{v}^\dagger_{M,op} \cdot \vec{v}_{M,op})^{-1/2} \ ,$$
$$m \cdot \gamma_{M,op} \cdot \vec{v}_{M,op} \equiv \ kinetic \ momentum \ vector \ \equiv \ mechanical \ momentum \ vector \ . \tag{0.5}$$

Here $\phi_{M,op}, \vec{A}_{M,op}, \vec{E}_{M,op}, \vec{B}_{M,op}$ represent the electromagnetic field, e, m denote charge and (non-vanishing) rest mass of the particle whereas g denotes the gyromagnetic factor [a]. The operators $\beta, \frac{2}{\hbar} \cdot \vec{\sigma}_{op}$ are Dirac matrices. This semiclassical Hamiltonian can be obtained from the Dirac Hamiltonian (modified by the Pauli term) by a certain relativistic generalization of the Foldy-Wouthuysen transformation [b] in which terms of second and higher order in the spin are dropped. The orbital variables $\vec{r}_{M,op}, \vec{p}_{M,op}$ and the spin variable $\vec{\sigma}_{op}$ obey the usual commutation relations. [2] The operator $\vec{r}_{M,op}$ is often called the operator of 'canonical mean position',

---

[1] I choose units where the vacuum light velocity is $c = 1$.

[2] Specifically $\vec{r}_{M,op}, \vec{p}_{M,op}$ obey the canonical commutation relations whereas $2 \cdot \vec{\sigma}_{op}/\hbar$ obeys the same commutation relations as the Pauli matrices.

which explains the presence of the subscript 'M'. Note also that for vanishing electromagnetic field $\vec{r}_{M,op}$ is the position operator of Newton and Wigner [NW49]. The operator $\vec{\sigma}_{op}$ defines the spin in a frame at rest w.r.t. the canonical mean position. The spin part $H_{M,op,spin}$ of the semiclassical Hamiltonian is in the form of a Stern-Gerlach (SG) energy term and the orbital part $H_{M,op,orb}$ resembles the standard orbital Hamiltonian. With this semiclassical Hamiltonian the Heisenberg equations of motion for the orbital operators comprise the Lorentz force and SG like terms. The Heisenberg equation of motion for the spin operator is equivalent to the Thomas-Bargmann-Michel-Telegdi equation (Thomas-BMT equation) [BMT59, Tho27]. This semiclassical Hamiltonian appears in the calculation of natural radiative polarization of electrons (the Sokolov-Ternov effect) [DK73, Jac76, ST64] in inhomogeneous fields. However, the interpretation of this Hamiltonian in terms of classical variables is also of great utility and has been used to describe the SG force for relativistic particles as well as to construct the action-angle variables for classical spin-orbit motion [BHR94a, BHR94b, DK73, Der90a, Der90b, Yok87]. Since from the outset the semiclassical Hamiltonian is to be applied in a context where $\hbar^2$ can be neglected, the classical version contains the same information without additional approximation. So it would be an unnecessary complication to continue to use the quantum version – one can just view the latter as a catalyst. Thus in the remainder of this paper I will use the classical form.

My interest in relativistic SG forces stems from the suggestion that they could be used to separate spin states in storage rings and thereby provide polarized (anti-) proton beams [CPP95, NR87]. Clearly, since the SG force is very small, one must be careful to include *all* terms in calculations. This is especially true in storage ring physics where there are not only strong transverse magnetic field gradients but also strong longitudinal gradients as well as large high frequency electric fields. One way to include all terms automatically is to use a formulation based on a Hamiltonian and of course one immediately thinks of the Derbenev-Kondratenko (DK) Hamiltonian. A formulation based on a Hamiltonian is also symplectic. The Hamiltonian approach has already been developed in [BHR94a, BHR94b].

I have opened this paper with a review of the DK approach to spin and orbital motion but there is a large literature on classical, Poincaré covariant equations of motion in the presence of SG forces going back as far as the paper of Frenkel of 1926 [Fre26]. See the reviews in [BT80, Nyb62, Pla66b, Roh72, TVW80]. So, naturally, given the prominence of the DK Hamiltonian in the handling of spin effects in storage rings, one would like to know how it is related to other formalisms and whether it leads to Poincaré covariant equations of motion. One would also like to know what general form the covariant equations of spin-orbit motion can take and to study the numerical implications of various choices.

Although I also treat some other topics the main burden of this paper is then the following:

- I prove that the equations of motion derived from the DK Hamiltonian are, after a transformation of the coordinates, identical to those parts of the Frenkel equations [Fre26] obtained after dropping second and higher order terms in the spin, and are therefore Poincaré covariant. To the knowledge of the author the present proof is the first and it occupies most of the present work. [3] The key to the proof is the realization that the position variable of the DK equations is not the spatial part of a space-time position but that this can be remedied if one first transforms it into the position variable given by Pryce which does not have this disadvantage. [4] This leads to the conclusion that

---

[3] For the special case of the first order SG force a proof was effectively contained in [DS70].

[4] I learnt this 'trick' from [DS70] who applied it to the special case of the first order SG force.

the approach of [DK73] is consistent with special relativity [c]. Thus the treatments in [BHR94a, BHR94b, Der90a, Der90b] are also relativistic.

- I then demonstrate, at first order in spin, how to construct more general classical Poincaré covariant equations of spin-orbit motion and find that there is an unlimited number of possibilities allowed by the kinematic constraints with each possibility being characterized by the choice of five parameters. This unified approach enables one to classify equations appearing in the literature by identifying their characteristic parameters. In particular, besides the Frenkel equations, three other cases are considered in more detail: the Good-Nyborg-Rafanelli (=GNR) equations [Goo62, Nyb64, Raf70] (which is also the approach chosen in [CPP95, NR87]), a case dealt with in [CoM94], a case dealt with in [Cos94], and the simple case of vanishing SG force. This technique could be extended to higher order in spin and the number of characteristic parameters would then increase.

- Having achieved my main aims I then investigate in more detail another relationship between the DK equations and some of the other spin-orbit systems and show that some spin-orbit systems can be related by transformations. In particular I relate the Frenkel equations to the GNR equations.

- I obtain numerical estimates of the SG force for simple spin and field configurations in the HERA proton storage ring [Br95] and in traps and compare the expectations from various equations.

Even if one has a basis for choosing among the plethora of spin-orbit systems available, for example by experiment or by appealing to the Dirac equation, one must still decide which variables to use for representing the particle position. The natural choice is the Pryce position variable since it transforms like the spatial part of the space-time position. On the other hand, in Newton-Wigner coordinates the DK equations are in canonical form. As we shall see later, Newton-Wigner coordinates can lead to equations of motion very different from those for the Pryce coordinates. However, the Newton-Wigner coordinate differs by less than the Compton wave length from the Pryce coordinate, so that the relationship between the Frenkel and DK equations is not just mathematical but expresses the effective physical identity of the DK particle and the Frenkel particle.

The paper is organized as follows.. In sections 1 and 2 the spin-orbit system defined by the DK equations is considered. In sections 3 and 4 I consider the spin-orbit system defined by the Frenkel equations and I show that it is the same system as in sections 1 and 2. The central part of the work is section 5 where the most general form for the spin-orbit systems under study is defined (equations (5.5)) and their main properties are explored. In sections 6,7,8,9 I simply expose some further properties and give numerical examples. In particular in section 8 I consider the SG force in magnetic fields typical for HERA-p and in section 9 the SG force in electromagnetic traps. In section 10 I introduce the transformations between different spin-orbit systems, mentioned above.

Throughout this work the time evolution of the spin is always that of [BMT59, Tho27] and the orbital motion is determined by the Lorentz force plus the SG force. The SG force consists of two parts, a part linear in the electromagnetic field vectors (the 'first order SG force') and a part quadratic in these fields (the 'second order SG force').

1 The DK equations

1.1 The DK Hamiltonian

The classical DK Hamiltonian $[d]$ is obtained from the semiclassical version (0.1) by replacing the operators by classical canonical variables and the commutators by the following Poisson bracket relations [5] $[e]$:

$$\{r_{M,j}, p_{M,k}\}_M = \delta_{jk} \;, \qquad \{r_{M,j}, r_{M,k}\}_M = \{p_{M,j}, p_{M,k}\}_M = \{r_{M,j}, \sigma_k\}_M = \{p_{M,j}, \sigma_k\}_M = 0 \;,$$

$$\{\sigma_j, \sigma_k\}_M = \sum_{m=1}^{3} \varepsilon_{jkm} \cdot \sigma_m \;. \qquad (j,k = 1,2,3) \tag{1.1}$$

Note that $\vec{r}_M, t$ are the position and time variables used in the everyday business of accelerator physics if calculations are made in 'cartesian coordinates'. The electromagnetic field is characterized either by the potentials $\phi_M, \vec{A}_M$ or the field vectors $\vec{E}_M, \vec{B}_M$. Thus one has $[f]$:

$$\vec{B}_M = \vec{\nabla}_M \wedge \vec{A}_M \;, \qquad \vec{E}_M = -\vec{\nabla}_M \phi_M - \frac{\partial \vec{A}_M}{\partial t} \;. \tag{1.2}$$

The functions $\phi_M, \vec{A}_M, \vec{E}_M, \vec{B}_M$ depend on $\vec{r}_M, t$. This also fixes the meaning of the partial derivative $\partial/\partial t$ in (1.2). The corresponding Maxwell equations are given in section 2. On introducing the quantities $[g]$:

$$\vec{\pi}_M \equiv \vec{p}_M - e \cdot \vec{A}_M \;,$$
$$J_M \equiv \sqrt{\vec{\pi}_M^\dagger \cdot \vec{\pi}_M + m^2} \;,$$
$$\vec{W}_M \equiv -\frac{e}{m} \cdot \left([\frac{m}{J_M} + \frac{g-2}{2}] \cdot \vec{B}_M - \frac{g-2}{2} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M \right.$$
$$\left. - [\frac{g}{2 \cdot J_M} - \frac{1}{J_M + m}] \cdot (\vec{\pi}_M \wedge \vec{E}_M) \right) \;, \tag{1.3}$$

the classical version of the DK Hamiltonian reads as [DK73]:

$$H_M(\vec{r}_M, \vec{p}_M, \vec{\sigma}, t) \equiv J_M + e \cdot \phi_M + \vec{\sigma}^\dagger \cdot \vec{W}_M \;. \tag{1.4}$$

To maintain consistency with the semiclassical nature of the DK Hamiltonian, throughout this paper all terms of second and higher order in the spin are neglected. This is the underlying approximation used in this paper. In particular one then always obtains the spin motion given by [BMT59, Tho27]. [6]

1.2 The DK equations

With this Hamiltonian the equations of motion for the orbital and spin variables - the DK equations - can be obtained using (1.1) and are: [7]

$$\vec{r}_M' = \{\vec{r}_M, H_M\}_M \;, \tag{1.5a}$$

---

[5]The variables $\vec{r}_M, \vec{p}_M, \vec{\sigma}$ together with the Poisson bracket $\{\ ,\ \}_M$ define a Poisson algebra for the DK Hamiltonian.

[6]This approximation is standard also in applications to accelerator physics. For the meaning of a power series expansion in spin, see also [Pla66b].

[7]In this paper the total time derivative is denoted by the prime '.

$$\vec{p}\,'_M = \{\vec{p}_M, H_M\}_M = \frac{e}{J_M} \cdot (\vec{\pi}_M \wedge \vec{B}_M) - e \cdot \vec{\nabla}_M \phi_M + \frac{e}{J_M} \cdot (\vec{\pi}_M^\dagger \cdot \vec{\nabla}_M) \vec{A}_M - \underbrace{\vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{W}_M)}_{\text{SG term}},$$

$$\tag{1.5b}$$

$$\vec{\sigma}\,' = \{\vec{\sigma}, H_M\}_M = \vec{W}_M \wedge \vec{\sigma}\,, \tag{1.5c}$$

where the SG terms in the orbital equations have been explicitly identified. The covariance of (1.5) under the Poincaré group will be demonstrated in section 5. Note that (1.5c) is equivalent to the Thomas-BMT equation [BMT59, Tho27].

In the following it will be useful to define:

$$\vec{v}_M \equiv \vec{r}\,'_M\,, \tag{1.6a}$$

$$\gamma_M \equiv (1 - \vec{v}_M^\dagger \cdot \vec{v}_M)^{-1/2}\,, \tag{1.6b}$$

$$\vec{\Omega}_M \equiv -\frac{e}{m} \cdot \left([\frac{1}{\gamma_M} + \frac{g-2}{2}] \cdot \vec{B}_M - \frac{g-2}{2} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \right.$$
$$\left. - [\frac{g}{2} - \frac{\gamma_M}{\gamma_M + 1}] \cdot (\vec{v}_M \wedge \vec{E}_M) \right). \tag{1.6c}$$

By neglecting the spin terms in (1.5-6) one gets the familiar relations:

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = e \cdot (\vec{v}_M \wedge \vec{B}_M) + e \cdot \vec{E}_M\,, \qquad \gamma'_M = \frac{e}{m} \cdot \vec{v}_M^\dagger \cdot \vec{E}_M\,.$$

With (1.6) I find that (1.5c) reads as:

$$\vec{\sigma}\,' = \vec{\Omega}_M \wedge \vec{\sigma}\,. \tag{1.7}$$

This is the usual form of the Thomas-BMT equation [BMT59, Tho27]. The transformation properties of $\vec{r}_M, t, \vec{v}_M, \vec{\sigma}$ under the Poincaré group are discussed in detail in section 5.

1.3

By (1.1),(1.4),(1.5a) I have: [8]

$$\vec{v}_M = \{\vec{r}_M, H_M\}_M = \frac{\vec{\pi}_M}{J_M} + \underbrace{\frac{e}{J_M^3} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M + \frac{e}{m \cdot J_M \cdot (J_M + m)^2} \cdot \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \vec{\pi}_M}_{\text{SG terms}}$$

$$\underbrace{- \frac{e}{m \cdot (J_M + m)} \cdot (\vec{E}_M \wedge \vec{\sigma}) + \frac{e \cdot g}{2 \cdot m \cdot J_M} \cdot (\vec{E}_M \wedge \vec{\sigma}) - \frac{e \cdot g}{2 \cdot m \cdot J_M^3} \cdot \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \vec{\pi}_M}_{\text{SG terms}}$$

$$+ \frac{g-2}{2} \cdot \left(\underbrace{\frac{e}{m} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{\sigma} \cdot \vec{B}_M}_{\text{SG term}} \right.$$

$$\left. \underbrace{- \frac{e}{m} \cdot \frac{m + 2 \cdot J_M}{J_M^3 \cdot (J_M + m)^2} \cdot \vec{\pi}_M^\dagger \cdot \vec{\sigma} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M + \frac{e}{m} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma}}_{\text{SG terms}} \right). \tag{1.8}$$

[8]This equation is derived in Appendix B, see: (B.3-4).

Thus for the 'M' variables the canonical momentum is different from the kinetic momentum, i.e.

$$\vec{\pi}_M \neq m \cdot \gamma_M \cdot \vec{v}_M \ ,$$

which says that in the 'M' variables one has zitterbewegung. This effect disappears with the electromagnetic field. [9]

The spin vector is constrained by:

$$\vec{\sigma}^\dagger \cdot \vec{\sigma} = \hbar^2/4 \ . \tag{1.9}$$

Because (1.9) is of second order in spin it plays no role in this paper. It is only applied in sections 8 and 9 for numerical calculations, where it gives the spin vector its correct length. Hence (1.9) acts as a numerical constraint to be inserted if the formulae are numerically evaluated. Note also that (1.9) is conserved under (1.5c).

My next task will be to rewrite equations (1.5) in a form which facilitates comparison with other formalisms.

2 Reexpressing the DK equations in terms of auxiliary variables

2.1

In this section I replace the variables $\vec{r}_M, \vec{v}_M, \vec{\sigma}$ by new variables $\vec{r}_P, \vec{v}_P, \vec{s}$ which will later serve as the building blocks of the Poincaré covariant formulae to be derived.

I define

$$\vec{r}_P \equiv \vec{r}_M + \frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M) \ . \tag{2.1}$$

The quantity $\vec{r}_P$ is the position variable of Pryce [Pry49]. The corresponding operator $\vec{r}_{P,op}$ describes (as $\vec{r}_{M,op}$ does) a particle which is not pointlike from the point of view of the Dirac position operator. For the explicit form of these operators in the Dirac representation, see [Hei]. For the special case of the first order SG force, see for example: [DS70, DS72]. Note that $\vec{r}_M$ and $\vec{r}_P$ differ by less than the Compton wave length. In contrast to $\vec{r}_M$, the variable $\vec{r}_P$ transforms under Poincaré transformations as the spatial part of the space-time position (see section 3). [10] In the present paper, where I work classically, I do not rely on the Dirac equation but instead derive the Poincaré covariance of the DK equations from the covariance property of $\vec{r}_P$.

The components of $\vec{r}_P$ are not canonical, i.e.

$$\{r_{P,j}, r_{P,k}\}_M = \frac{1}{m^2} \cdot \sum_{l=1}^3 \varepsilon_{jkl} \cdot \left(\frac{1}{\gamma_M^2} \cdot s_l + v_{M,l} \cdot \vec{v}_M^\dagger \cdot \vec{s}\right) \neq 0 \ . \qquad (j, k = 1, 2, 3) \tag{2.2}$$

---

[9]The term 'zitterbewegung' was introduced in [Sch30] and in its original sense it only applies to a free quantum mechanical particle. Thus my term deals with the classical analogue and even applies in the presence of electromagnetic fields. For more details, see [Cor68, Fol62].

[10]The quantum mechanical proof of this covariance property, based on the Dirac equation (plus Pauli term), deals with the operator $\vec{r}_{P,op}$ and is given for the special case of the first order SG force in [DS72]. For the general proof, see [Hei].

But the complementary virtues of my two position variables are now clear; the variable $\vec{r}_P$ is useful for studying covariance of the equations of motion (as seen below) whereas $\vec{r}_M$ is useful for symplectic calculus (calculating spin-orbit transport maps) because the Poisson brackets for $\vec{r}_M$ are canonical (see (1.1)).

The velocity vector corresponding to $\vec{r}_P$ reads as:

$$\vec{v}_P \equiv \vec{r}_P' = \vec{v}_M + \left(\frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M) \right)' , \qquad (2.3a)$$

and I define:

$$\gamma_P \equiv (1 - \vec{v}_P^\dagger \cdot \vec{v}_P)^{-1/2} . \qquad (2.3b)$$

Furthermore I define:

$$\vec{s} \equiv \gamma_M \cdot \vec{\sigma} - \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M , \qquad (2.4)$$

from which follows:

$$\vec{\sigma} = \frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P ,$$

$$\vec{v}_P^\dagger \cdot \vec{\sigma} = \vec{v}_P^\dagger \cdot \vec{s} ,$$

$$\{s_j, s_k\}_M = \sum_{l=1}^{3} \varepsilon_{jkl} \cdot (s_l + \gamma_P^2 \cdot v_{P,l} \cdot \vec{v}_P^\dagger \cdot \vec{s}) . \qquad (j, k = 1, 2, 3) \qquad (2.5)$$

One can therefore express the evolutions of $\vec{r}_M(t), \vec{v}_M(t), \vec{\sigma}(t)$ in terms of $\vec{r}_P(t), \vec{v}_P(t), \vec{s}(t)$. The field equations for the 'M' fields are

$$0 = \vec{\nabla}_M \wedge \vec{B}_M - \frac{\partial \vec{E}_M}{\partial t} , \qquad 0 = \vec{\nabla}_M \wedge \vec{E}_M + \frac{\partial \vec{B}_M}{\partial t} , \qquad 0 = \vec{\nabla}_M^\dagger \cdot \vec{E}_M , \qquad 0 = \vec{\nabla}_M^\dagger \cdot \vec{B}_M ,$$
$$(2.6)$$

which are the vacuum Maxwell equations. On introducing the abbreviations

$$\vec{E}_P \equiv \vec{E}_M(\vec{r}_P, t) , \qquad \vec{B}_P \equiv \vec{B}_M(\vec{r}_P, t) , \qquad (2.7)$$

one then obtains [h]:

$$0 = \vec{\nabla}_P \wedge \vec{B}_P - \frac{\partial \vec{E}_P}{\partial t} , \qquad 0 = \vec{\nabla}_P \wedge \vec{E}_P + \frac{\partial \vec{B}_P}{\partial t} , \qquad 0 = \vec{\nabla}_P^\dagger \cdot \vec{E}_P , \qquad 0 = \vec{\nabla}_P^\dagger \cdot \vec{B}_P ,$$
$$(2.8)$$

which also are the vacuum Maxwell equations. The transformation properties of $\vec{r}_P, t, \vec{v}_P, \vec{s}, \vec{B}_P, \vec{E}_P$ under the Poincaré group are discussed in detail in section 5.

2.2

In Appendix A it is shown that the DK equations (1.5) lead by (2.4) to:

$$\vec{s}\,' = \frac{e}{m} \cdot [\frac{g-2}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot (\vec{E}_P + \vec{v}_P \wedge \vec{B}_P) + \frac{1}{\gamma_P} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}$$

$$- \frac{g-2}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P + \frac{g}{2} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{B}_P) - \frac{g}{2} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P] . \qquad (2.9)$$

9

2.3

Moreover in Appendix B it is shown that the DK equations (1.5) lead by (2.3) to:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m} \cdot [2 \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P - g \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P + (g-2) \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+ (g-2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P' - (g-2) \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{v}_P \wedge \vec{s})]$$

$$+ \frac{e^2}{4 \cdot m^2} \cdot [-(g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) + (g-2)^2 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$- (g-2)^2 \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P + (-g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 4) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- (g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) + (g-2)^2 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+ (g-2)^2 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - (g^2 - 4 \cdot g) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+ (g-2) \cdot g \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+ (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- (g^2 - 4 \cdot g) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot g \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] . \qquad (2.10)$$

2.4

On collecting (2.3a),(2.9-10) one finds that the DK equations (1.5) are equivalent to:

$$\vec{r}_P' = \vec{v}_P , \qquad (2.11a)$$

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m} \cdot [2 \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P - g \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P + (g-2) \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+ (g-2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P' - (g-2) \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{v}_P \wedge \vec{s})]$$

$$+ \frac{e^2}{4 \cdot m^2} \cdot [-(g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) + (g-2)^2 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$- (g-2)^2 \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P + (-g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 4) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- (g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) + (g-2)^2 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+ (g-2)^2 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - (g^2 - 4 \cdot g) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+ (g-2) \cdot g \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+ (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- (g^2 - 4 \cdot g) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot g \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] , \qquad (2.11b)$$

$$\vec{s}\,' = \frac{e}{m} \cdot [\frac{g-2}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot (\vec{E}_P + \vec{v}_P \wedge \vec{B}_P) + \frac{1}{\gamma_P} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}$$

$$- \frac{g-2}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P + \frac{g}{2} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{B}_P) - \frac{g}{2} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P] . \qquad (2.11c)$$

The covariance of (2.11) under the Poincaré group will be demonstrated in section 5. By

neglecting the spin terms in (2.11b) one gets the familiar relations:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P \, , \qquad \gamma'_P = \frac{e}{m} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \, . \tag{2.12}$$

3 The equivalence between the DK equations and the Frenkel equations

3.1

In this section the DK equations are shown to be equivalent to the Frenkel equations.

To study the covariance properties of $\vec{r}_P, \vec{v}_P, \vec{s}, t$ I first construct the variables X^P, U^P where the components X_μ^P are defined by [i]:

$$X_\mu^P = (\vec{r}_P^\dagger, i \cdot t)_\mu \, , \qquad (\mu = 1, ..., 4) \tag{3.1}$$

and represent the space-time position. The components U_μ^P denote the corresponding 4-velocity, i.e. [11]

$$U_\mu^P = \dot{X}_\mu^P \equiv \frac{d}{d\tau} X_\mu^P = \gamma_P \cdot \frac{d}{dt} X_\mu^P = (\gamma_P \cdot \vec{v}_P^\dagger, i \cdot \gamma_P)_\mu \, , \qquad (\mu = 1, ..., 4) \tag{3.2}$$

where τ denotes the proper time.

The utility of the spin vector $\vec{s}$ becomes apparent by encoding it in a spin tensor S^P. To achieve this I first introduce the 'dipole moment tensor' [j]:

$$M_{\mu\nu}^P = \begin{pmatrix} 0 & \mu_{P,3} & -\mu_{P,2} & i \cdot \varepsilon_{P,1} \\ -\mu_{P,3} & 0 & \mu_{P,1} & i \cdot \varepsilon_{P,2} \\ \mu_{P,2} & -\mu_{P,1} & 0 & i \cdot \varepsilon_{P,3} \\ -i \cdot \varepsilon_{P,1} & -i \cdot \varepsilon_{P,2} & -i \cdot \varepsilon_{P,3} & 0 \end{pmatrix}_{\mu\nu} \, , \qquad (\mu, \nu = 1, ..., 4) \tag{3.3}$$

which by definition transforms as a tensor of rank 2. One calls $\vec{\mu}_P$ the 'magnetic dipole moment' and $\vec{\varepsilon}_P$ the 'electric dipole moment' [Nyb64]. The magnetic dipole moment resp. the electric dipole moment in the rest frame is denoted by $\vec{\mu}_R$ resp. $\vec{\varepsilon}_R$ and by definition it is the intrinsic magnetic dipole moment resp. intrinsic electric dipole moment of the particle. The tensor M^P appears frequently in the literature on the relativistic SG force and it occurs already in the 1926 paper of Frenkel [Fre26]. Note that M^P is also used in the theory of relativistic fluids resp. composite particles. [12]

Although M^P is especially useful for treating particles with combined intrinsic magnetic and intrinsic electric dipole moments, such particles are not treated in this paper. In this study I only consider particles without intrinsic electric dipole moment. Denoting the dipole moment tensor in the rest frame by M^R, I then have:

$$M_{\mu\nu}^R = \begin{pmatrix} 0 & \mu_{R,3} & -\mu_{R,2} & 0 \\ -\mu_{R,3} & 0 & \mu_{R,1} & 0 \\ \mu_{R,2} & -\mu_{R,1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} \, . \qquad (\mu, \nu = 1, ..., 4)$$

---

[11]In this paper the upper dot symbol denotes the total proper time derivative.

[12]See for example the textbook treatments in [DS72, Moe72, Syn58].

Bearing in mind that for a charged particle the intrinsic magnetic dipole moment is related to the rest frame spin by:

$$\vec{\mu}_R \equiv \frac{e \cdot g}{2 \cdot m} \cdot \vec{\sigma} \, , \tag{3.4}$$

one obtains:

$$M^R_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 & 0 \\ -\sigma_3 & 0 & \sigma_1 & 0 \\ \sigma_2 & -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} . \qquad (\mu, \nu = 1, ..., 4)$$

One now defines the spin tensor S^P as that tensor, which in the rest frame reads as:

$$S^R_{\mu\nu} = \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 & 0 \\ -\sigma_3 & 0 & \sigma_1 & 0 \\ \sigma_2 & -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} . \qquad (\mu, \nu = 1, ..., 4) \tag{3.5}$$

Thus the spin tensor S^P is given by:

$$M^P = \frac{e \cdot g}{2 \cdot m} \cdot S^P \, .$$

By transforming back from the rest frame (see subsection 7.2) one obtains: [13]

$$S^P_{\mu\nu} = \begin{pmatrix} 0 & s_3 & -s_2 & -i \cdot q_1 \\ -s_3 & 0 & s_1 & -i \cdot q_2 \\ s_2 & -s_1 & 0 & -i \cdot q_3 \\ i \cdot q_1 & i \cdot q_2 & i \cdot q_3 & 0 \end{pmatrix}_{\mu\nu} , \qquad \vec{q} = \vec{s} \wedge \vec{v}_P \, , \qquad (\mu, \nu = 1, ..., 4) \tag{3.6}$$

so that one has

$$\vec{\mu}_P \equiv \frac{e \cdot g}{2 \cdot m} \cdot \vec{s} \, , \qquad \vec{\varepsilon}_P \equiv \frac{e \cdot g}{2 \cdot m} \cdot (\vec{v}_P \wedge \vec{s}) \, . \tag{3.7}$$

For a particle with vanishing intrinsic electric dipole moment equations (3.2),(3.6) enshrine the kinematic constraints [k]:

$$U^P_\mu \cdot U^P_\mu = -1 \, , \tag{3.8a}$$

$$S^P_{\mu\nu} \cdot U^P_\nu = 0 \, . \qquad (\mu = 1, ..., 4) \tag{3.8b}$$

With (3.6) I have encoded the spin vector $\vec{s}$ in the spin tensor S^P. For the consistency of the rank 2 tensor property of S^P with the transformation properties of the spin vector $\vec{s}$ and of $\vec{v}_P$, see subsections 5.4 and 5.5.

I also introduce the rank two antisymmetric tensor field describing the electromagnetic field and whose components are defined by

$$F^P_{\mu\nu} \equiv \begin{pmatrix} 0 & B_{P,3} & -B_{P,2} & -i \cdot E_{P,1} \\ -B_{P,3} & 0 & B_{P,1} & -i \cdot E_{P,2} \\ B_{P,2} & -B_{P,1} & 0 & -i \cdot E_{P,3} \\ i \cdot E_{P,1} & i \cdot E_{P,2} & i \cdot E_{P,3} & 0 \end{pmatrix}_{\mu\nu} , \qquad (\mu, \nu = 1, ..., 4) \tag{3.9}$$

---

[13] The corresponding operator $S^{P,op}$ is obtained in [FG61a, HW63].

i.e.

$$F^P \leftrightarrow (\vec{B}_P, -i \cdot \vec{E}_P) \ .$$

Then [l] with the definition:

$$\partial_\mu^P \equiv (\frac{\partial}{\partial X_1^P} \ , \ \frac{\partial}{\partial X_2^P} \ , \ \frac{\partial}{\partial X_3^P} \ , \ \frac{\partial}{\partial X_4^P})_\mu \ , \qquad (\mu = 1, ..., 4) \tag{3.10}$$

the vacuum Maxwell equations (2.8) read as:

$$0 = \partial_\rho^P F_{\mu\nu}^P + \partial_\mu^P F_{\nu\rho}^P + \partial_\nu^P F_{\rho\mu}^P \ , \qquad 0 = \partial_\mu^P F_{\mu\nu}^P \ . \qquad (\mu, \nu, \rho = 1, ..., 4) \tag{3.11}$$

3.2

I can now reexpress the evolution (2.11) of $\vec{r}_P(t), \vec{v}_P(t), \vec{s}(t)$ in terms of $X^P(\tau), U^P(\tau), S^P(\tau)$. In subsection 5.2 it is shown that equations (2.11) are equivalent to:

$$\dot{X}_\mu^P = U_\mu^P \ , \tag{3.12a}$$

$$\dot{U}_\mu^P = \frac{e}{m} \cdot F_{\mu\nu}^P \cdot U_\nu^P - \frac{e \cdot g}{4 \cdot m^2} \cdot \left(S_{\nu\omega}^P \cdot \partial_\mu^P F_{\omega\nu}^P + U_\mu^P \cdot S_{\nu\omega}^P \cdot U_\lambda^P \cdot \partial_\lambda^P F_{\omega\nu}^P \right)$$

$$+ \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot S_{\mu\nu}^P \cdot U_\omega^P \cdot U_\lambda^P \cdot \partial_\lambda^P F_{\nu\omega}^P - \frac{e^2 \cdot (g-2)^2}{4 \cdot m^3} \cdot S_{\mu\nu}^P \cdot F_{\nu\omega}^P \cdot F_{\omega\rho}^P \cdot U_\rho^P$$

$$+ \frac{e^2 \cdot (g-2) \cdot g}{4 \cdot m^3} \cdot F_{\mu\nu}^P \cdot S_{\nu\omega}^P \cdot F_{\omega\rho}^P \cdot U_\rho^P - \frac{e^2 \cdot g}{4 \cdot m^3} \cdot F_{\mu\nu}^P \cdot U_\nu^P \cdot F_{\lambda\omega}^P \cdot S_{\omega\lambda}^P \ , \tag{3.12b}$$

$$\dot{S}_{\mu\nu}^P = \frac{e \cdot g}{2 \cdot m} \cdot \left(F_{\mu\omega}^P \cdot S_{\omega\nu}^P - S_{\mu\omega}^P \cdot F_{\omega\nu}^P \right)$$

$$- \frac{e \cdot (g-2)}{2 \cdot m} \cdot \left(S_{\mu\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\nu^P - S_{\nu\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\mu^P \right) \ . \qquad (\mu, \nu = 1, ..., 4) \tag{3.12c}$$

These are the equations given by Frenkel in 1926 [Fre26]. [14] They respect the kinematic constraints (3.8) and I can therefore conclude that:

- The DK equations are equivalent to the Frenkel equations.

In this derivation it was essential that the electromagnetic field obeys the vacuum Maxwell equations. The covariance of (3.12) under the Poincaré group will be demonstrated in section 5. Note also that (3.12c) is equivalent to the BMT equation [BMT59, Nyb64]. The term

$$\frac{e \cdot g}{2 \cdot m} \cdot \left(F_{\mu\omega}^P \cdot S_{\omega\nu}^P - S_{\mu\omega}^P \cdot F_{\omega\nu}^P - S_{\mu\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\nu^P + S_{\nu\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\mu^P \right)$$

is that part of the rhs of (3.12c) which is independent of the forces acting on the orbital motion. The remaining term

$$\frac{e}{m} \cdot \left(S_{\mu\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\nu^P - S_{\nu\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\mu^P \right)$$

represents the Thomas precession [Tho27].

It follows from the normalization of the spin vector that:

$$S_{\mu\nu}^P \cdot S_{\mu\nu}^P = \hbar^2/2 \ . \tag{3.13}$$

As in (1.9) this equation is of second order in spin so that it plays no role in this paper. Note also that (3.13) is conserved under (3.12c).

---

[14] Equations (3.12) are equivalent to equations (13a),(14),(21), (21a-b) in [Fre26]. They were rederived by many authors. See also the reviews in [BT80, Nyb62, Pla66b, Roh72, TVW80].

4 Rederiving the Frenkel equations in terms of a Hamiltonian which is a Poincare scalar

4.1

The reader will perhaps be interested to learn that one can rederive the Frenkel equations from a Hamiltonian with the proper time as independent variable.

One begins by noting that X^P, P^P and S^P obey the following Poisson bracket relations [Cor68]: [15]

$$\{X_\mu^P, P_\nu^P\}_P = \delta_{\mu\nu} ,$$
$$\{S_{\mu\nu}^P, S_{\lambda\omega}^P\}_P = S_{\mu\lambda}^P \cdot \delta_{\nu\omega} + S_{\nu\omega}^P \cdot \delta_{\mu\lambda} - S_{\mu\omega}^P \cdot \delta_{\nu\lambda} - S_{\nu\lambda}^P \cdot \delta_{\mu\omega} ,$$
$$\{X_\mu^P, X_\nu^P\}_P = \{X_\mu^P, S_{\nu\lambda}^P\}_P = \{P_\mu^P, P_\nu^P\}_P = \{P_\mu^P, S_{\nu\lambda}^P\}_P = 0 . \quad (\mu, \nu, \lambda, \omega = 1, ..., 4) \quad (4.1)$$

Then I introduce the abbreviations

$$M^P \equiv m + \frac{e \cdot g}{4 \cdot m} \cdot S_{\nu\omega}^P \cdot F_{\omega\nu}^P ,$$
$$A_\mu^P \equiv (\vec{A}_P^\dagger, i \cdot \phi_P)_\mu ,$$
$$\Pi_\mu^P \equiv P_\mu^P - e \cdot A_\mu^P \equiv (\vec{\pi}_P^\dagger, \Pi_4^P)_\mu , \qquad (\mu = 1, ..., 4) \quad (4.2)$$

where $\vec{A}_P(\vec{r}_P, t), \phi_P(\vec{r}_P, t)$ are the potentials of the electromagnetic field so that:

$$\vec{B}_P = \vec{\nabla}_P \wedge \vec{A}_P , \qquad \vec{E}_P = -\vec{\nabla}_P \phi_P - \frac{\partial \vec{A}_P}{\partial t} . \quad (4.3)$$

i.e:

$$F_{\mu\nu}^P = \partial_\mu^P A_\nu^P - \partial_\nu^P A_\mu^P . \qquad (\mu, \nu = 1, ..., 4) \quad (4.4)$$

The Hamiltonian is [m]:

$$H^P = (\frac{1}{m} - \frac{M^P}{2 \cdot m^2}) \cdot \Pi_\mu^P \cdot \Pi_\mu^P + \frac{1}{2} \cdot M^P + \frac{e \cdot (g-2)}{2 \cdot m^3} \cdot \Pi_\mu^P \cdot S_{\mu\nu}^P \cdot F_{\nu\omega}^P \cdot \Pi_\omega^P , \quad (4.5)$$

and the corresponding equations of motion are:

$$\dot{X}_\mu^P = \{X_\mu^P, H^P\}_P ,$$
$$\dot{U}_\mu^P = \{U_\mu^P, H^P\}_P = \{\{X_\mu^P, H^P\}_P, H^P\}_P ,$$
$$\dot{S}_{\mu\nu}^P = \{S_{\mu\nu}^P, H^P\}_P . \qquad (\mu, \nu = 1, ..., 4) \quad (4.6)$$

By evaluating the Poisson brackets one finds that these are the Frenkel equations (3.12).

Because S^P is a tensor of rank 2 and F^P is a tensor field of rank 2 it follows by (4.2) that M^P is a scalar field. Moreover, because P^P is a 4-vector and A^P a 4-vector field, one finds that all three terms of the Hamiltonian (4.5) are scalar fields.

---

[15]In the quantum mechanical analogue to the system (4.1) the position operator correponding to $\vec{r}_P$ must be different from $\vec{r}_{P,op}$ because this system involves (unlike the system for the 'M' variables) negative energy states. See for example [JM63, Cor68].

4.2

With (4.5) one gets:

$$\dot{\Pi}^P_\mu = \{\Pi^P_\mu, H^P\}_P = \frac{e}{m} \cdot F^P_{\mu\nu} \cdot \Pi^P_\nu - \frac{e \cdot g}{4 \cdot m} \cdot S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} + \frac{e^2 \cdot (g-2)}{2 \cdot m^3} \cdot F^P_{\mu\nu} \cdot S^P_{\nu\omega} \cdot F^P_{\omega\rho} \cdot \Pi^P_\rho$$

$$+ \frac{e^2 \cdot g}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot \Pi^P_\nu \cdot F^P_{\lambda\omega} \cdot S^P_{\omega\lambda} , \qquad (\mu = 1, ..., 4) \tag{4.7}$$

and:

$$U^P_\mu = (\frac{2}{m} - \frac{M^P}{m^2}) \cdot \Pi^P_\mu + \frac{e \cdot (g-2)}{2 \cdot m^3} \cdot S^P_{\mu\nu} \cdot F^P_{\nu\omega} \cdot \Pi^P_\omega . \qquad (\mu = 1, ..., 4) \tag{4.8}$$

With (4.8) one can write (4.7) in the elegant form:

$$\dot{\Pi}^P_\mu = e \cdot F^P_{\mu\nu} \cdot U^P_\nu - \frac{e \cdot g}{4 \cdot m} \cdot S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} , \qquad (\mu = 1, ..., 4) \tag{4.9}$$

which will be useful in subsection 7.3.

Using (4.8) the constraints (3.8), expressed in terms of the variables X^P, P^P and S^P, read as:

$$\Pi^P_\mu \cdot \Pi^P_\mu = m^2 - 2 \cdot m \cdot M^P ,$$
$$S^P_{\mu\nu} \cdot \Pi^P_\nu = 0 , \qquad (\mu = 1, ..., 4)$$

so that the Hamiltonian H^P vanishes, if the constraints are taken into account. One thus has a constrained Hamiltonian system. The constraints are to be taken into account only in the final results (e.g. the equations of motion).

4.3

In the 'P' variables the canonical momentum is $\vec{\pi}_P$ and using (4.2),(4.8) and Appendix D I get:

$$m \cdot \gamma_P \cdot \vec{v}_P = \vec{\pi}_P + \frac{e}{m^2} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{\pi}_P - \frac{e \cdot g}{2 \cdot m^3 \cdot \gamma_P} \cdot \vec{E}^\dagger_P \cdot (\vec{s} \wedge \vec{\pi}_P) \cdot \vec{\pi}_P$$

$$+ \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \vec{\pi}^\dagger_P \cdot \vec{s} \cdot \vec{B}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot (\vec{E}_P \wedge \vec{s}) .$$

I thus observe for the 'P' variables that the canonical momentum is different from the kinetic momentum, i.e.

$$\vec{\pi}_P \neq m \cdot \gamma_P \cdot \vec{v}_P .$$

As in case of the 'M' variables (see the end of section 1), one has zitterbewegung for the 'P' variables. This effects disappears with the electromagnetic field. [16]

---

[16]One can modify the Hamiltonian (4.5) in a way such that zitterbewegung arises even for the free particle. Then higher orders of the spin are important [Cor68, Pla66b].

5 Relating the Frenkel equations to other approaches. The Poincare covariance

5.1

The Frenkel equations are just a special case allowed by the kinematic constraints (3.8) and in fact various forms for the SG forces are possible even if one requires, as I do, that the spin equation is equivalent to the BMT equation and that the electromagnetic field obeys the vacuum Maxwell equations. [17] In fact the constraints (3.8) allow the following generalization of (3.12):

$$\dot{X}^P_\mu = U^P_\mu \,, \tag{5.1a}$$

$$\dot{U}^P_\mu = \frac{e}{m} \cdot F^P_{\mu\nu} \cdot U^P_\nu + Y^P_\mu \,, \tag{5.1b}$$

$$\dot{S}^P_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \left(F^P_{\mu\omega} \cdot S^P_{\omega\nu} - S^P_{\mu\omega} \cdot F^P_{\omega\nu} \right)$$

$$- \frac{e \cdot (g-2)}{2 \cdot m} \cdot \left(S^P_{\mu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\nu - S^P_{\nu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\mu \right) \,, \qquad (\mu, \nu = 1, ..., 4) \tag{5.1c}$$

where the 4-vector Y^P collects the SG force terms. I require that Y^P only depends on the following dimensional quantities: m, U^P, $e \cdot F^P$, S^P, ∂^P. If one assumes that the dependence on $U^P, e \cdot F^P, S^P, \partial^P$ is polynomial and in particular of first order in S^P, then by dimensional analysis the dependence on F^P resp. ∂^P is at most quadratic resp. linear and the constraints (3.8) lead to the following most general ansatz [n]:

$$Y^P_\mu = -\frac{e \cdot c_2}{4 \cdot m^2} \cdot \left(S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} + U^P_\mu \cdot S^P_{\nu\omega} \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\omega\nu} \right)$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot S^P_{\mu\nu} \cdot U^P_\omega \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\nu\omega} + \frac{e \cdot c_6}{2 \cdot m^2} \cdot \left(S^P_{\nu\omega} \cdot \partial^P_\omega F^P_{\mu\nu} + U^P_\mu \cdot S^P_{\lambda\omega} \cdot U^P_\nu \cdot \partial^P_\omega F^P_{\nu\lambda} \right)$$

$$+ \frac{e \cdot c_7}{2 \cdot m^2} \cdot S^P_{\mu\nu} \cdot \partial^P_\rho F^P_{\nu\rho} + \frac{e^2 \cdot c_3}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot S^P_{\nu\omega} \cdot F^P_{\omega\rho} \cdot U^P_\rho - \frac{e^2 \cdot c_4}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot U^P_\nu \cdot S^P_{\omega\rho} \cdot F^P_{\rho\omega}$$

$$+ \frac{e^2 \cdot c_5}{4 \cdot m^3} \cdot S^P_{\mu\nu} \cdot F^P_{\nu\rho} \cdot F^P_{\rho\omega} \cdot U^P_\omega \,, \qquad (\mu = 1, ..., 4) \tag{5.2}$$

where $c_1, ..., c_7$ are dimensionless real numbers. This can be further simplified by using the vacuum Maxwell equations (3.11), so that

$$S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} = -2 \cdot S^P_{\nu\omega} \cdot \partial^P_\nu F^P_{\mu\omega} \,. \qquad (\mu = 1, ..., 4) \tag{5.3}$$

Hence by (3.11) the terms on the rhs of (5.2) which are proportional to c_6, c_7 are not independent from the others so that the general ansatz (5.2) finally simplifies to:

$$Y^P_\mu = -\frac{e \cdot c_2}{4 \cdot m^2} \cdot \left(S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} + U^P_\mu \cdot S^P_{\nu\omega} \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\omega\nu} \right)$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot S^P_{\mu\nu} \cdot U^P_\omega \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\nu\omega} + \frac{e^2 \cdot c_3}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot S^P_{\nu\omega} \cdot F^P_{\omega\rho} \cdot U^P_\rho$$

$$- \frac{e^2 \cdot c_4}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot U^P_\nu \cdot S^P_{\omega\rho} \cdot F^P_{\rho\omega} + \frac{e^2 \cdot c_5}{4 \cdot m^3} \cdot S^P_{\mu\nu} \cdot F^P_{\nu\rho} \cdot F^P_{\rho\omega} \cdot U^P_\omega \,. \qquad (\mu = 1, ..., 4) \tag{5.4}$$

---

[17] An early work about the nonuniqueness of the relativistic SG force is: [Moe49].

Combining this with (5.1) I get the following general system of equations: [18]

$$\dot{X}^P_\mu = U^P_\mu \,, \tag{5.5a}$$

$$\dot{U}^P_\mu = \frac{e}{m} \cdot F^P_{\mu\nu} \cdot U^P_\nu - \frac{e \cdot c_2}{4 \cdot m^2} \cdot \left(S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} + U^P_\mu \cdot S^P_{\nu\omega} \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\omega\nu} \right)$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot S^P_{\mu\nu} \cdot U^P_\omega \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\nu\omega} + \frac{e^2 \cdot c_3}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot S^P_{\nu\omega} \cdot F^P_{\omega\rho} \cdot U^P_\rho$$

$$- \frac{e^2 \cdot c_4}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot U^P_\nu \cdot S^P_{\omega\rho} \cdot F^P_{\rho\omega} + \frac{e^2 \cdot c_5}{4 \cdot m^3} \cdot S^P_{\mu\nu} \cdot F^P_{\nu\rho} \cdot F^P_{\rho\omega} \cdot U^P_\omega \,, \tag{5.5b}$$

$$\dot{S}^P_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \left(F^P_{\mu\omega} \cdot S^P_{\omega\nu} - S^P_{\mu\omega} \cdot F^P_{\omega\nu} \right)$$

$$- \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \left(S^P_{\mu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\nu - S^P_{\nu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\mu \right) . \qquad (\mu, \nu = 1, ..., 4) \tag{5.5c}$$

The covariance of (5.5) under the Poincaré group will be demonstrated in subsection 5.3.

The Frenkel equations correspond to

$$c_1 = 0 \,, \qquad c_2 = g \,, \qquad c_3 = (g - 2) \cdot g \,, \qquad c_4 = g \,, \qquad c_5 = -(g - 2)^2 \,, \tag{5.6}$$

and by neglecting the second order SG terms in the Frenkel equations one gets the 'reduced' Frenkel equations, which correspond to:

$$c_1 = 0 \,, \qquad c_2 = g \,, \qquad c_3 = c_4 = c_5 = 0 \,. \tag{5.7}$$

The GNR equations, which are defined by [Goo62, Nyb64, Raf70], correspond to:

$$c_1 = g - 2 \,, \qquad c_2 = g \,, \qquad c_3 = c_4 = c_5 = 0 \,. \tag{5.8}$$

Another interesting choice is given by [Cos94]: [19]

$$c_1 = -2 \,, \qquad c_2 = g \,, \qquad c_3 = g^2 \,, \qquad c_4 = g \,, \qquad c_5 = -g^2 + 2 \cdot g \,. \tag{5.9}$$

This is a modification of a force given in [CoM94] by a 'redshift term': [20]

$$c_1 = -2 \,, \qquad c_2 = g \,, \qquad c_3 = g^2 \,, \qquad c_4 = 0 \,, \qquad c_5 = -g^2 \,. \tag{5.10}$$

A very simple choice is

$$c_1 = -2 \,, \qquad c_2 = c_3 = c_4 = c_5 = 0 \,, \tag{5.11}$$

corresponding to a SG force, which vanishes for the 'P' variables.

That the kinematic constraints (3.8) allow the general form (5.4) of Y^P illustrates that the five parameters $c_1, ..., c_5$ are, on the classical level, just phenomenological constants which can only be fixed by comparison with experiments (e.g. in storage rings with polarized beams) [o]. The physical implications of this plethora of possibilities will be addressed later (see sections 8,9). Note that in the above derivation of Y^P the assumption of first order dependence on spin was essential. The reader who is interested in higher order spin terms is advised to consult the large literature on classical relativistic spin. See, for example, the book [Cor68] for references. A recent interesting treatment, nonlinear in spin, can be found in [Cos94].

---

[18] Note also that for every choice of the characteristic parameters $c_1, ..., c_5$ the following equations of the previous sections remain valid: (1.1-4),(1.5c),(1.6-7),(1.9),(2.1-9),(2.11a),(2.11c),(3.1-11),(3.13).

[19] See equation [4.57] therein. Note that I neglect second order spin terms.

[20] See equation [15] in [CoM94]. Note that I neglect second order spin terms.

5.2

It follows from (5.5b) and by using Appendix D that:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \frac{e \cdot c_2}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m} \cdot [(c_1 + 2) \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P - c_2 \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P + (c_2 - c_1 - 2) \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+ (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P' - (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{v}_P \wedge \vec{s})]$$

$$+ \frac{e^2}{4 \cdot m^2} \cdot [c_5 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - c_5 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) + c_5 \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P$$

$$+ (-c_3 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P - c_5 + 2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P + c_5 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s})$$

$$- c_5 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - c_5 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+ (2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) + c_3 \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - c_3 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+ c_3 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + c_3 \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$+ (2 \cdot c_4 - c_3) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot c_4 \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] \; .$$

Collecting this with (2.3a),(2.9) I get

$$\vec{r}_P' = \vec{v}_P \; , \tag{5.12a}$$

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \frac{e \cdot c_2}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m} \cdot [(c_1 + 2) \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P - c_2 \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P + (c_2 - c_1 - 2) \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+ (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P' - (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{v}_P \wedge \vec{s})]$$

$$+ \frac{e^2}{4 \cdot m^2} \cdot [c_5 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - c_5 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) + c_5 \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P$$

$$+ (-c_3 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P - c_5 + 2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P + c_5 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s})$$

$$- c_5 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - c_5 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+ (2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) + c_3 \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - c_3 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+ c_3 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + c_3 \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$+ (2 \cdot c_4 - c_3) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot c_4 \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] \; , \tag{5.12b}$$

$$\vec{s}\,' = \frac{e}{m} \cdot [\frac{g-2}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot (\vec{E}_P + \vec{v}_P \wedge \vec{B}_P) + \frac{1}{\gamma_P} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}$$

$$- \frac{g-2}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P + \frac{g}{2} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{B}_P) - \frac{g}{2} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P] \; . \tag{5.12c}$$

It is shown in Appendix D that (5.12) is equivalent to (5.5). For the special choice (5.6) this means that (2.11) is equivalent to (3.12), i.e. that the DK equations are equivalent to the Frenkel equations!

5.3

In this subsection I show that equations (5.5) are Poincaré covariant. Note that X^P transforms as a space-time position, U^P as a 4-vector, S^P as a tensor of rank 2 and F^P as a tensor field

of rank 2. Thus (5.5) are covariant under the restricted Poincaré group $[p]$ because equations (5.5) are constructed from covariant quantities. To show that (5.5) is covariant under the whole Poincaré group I first consider space inversion (=parity transformation). This acts in the following way:

$$\tau \to \tau \,,$$
$$X_\mu^P \to (-\vec{r}_P^{\,\dagger}, X_4^P)_\mu \,,$$
$$U_\mu^P \to (-\gamma_P \cdot \vec{v}_P^{\,\dagger}, i \cdot \gamma_P)_\mu \,,$$
$$\partial_\mu^P \to (-\frac{\partial}{\partial X_1^P}, -\frac{\partial}{\partial X_2^P}, -\frac{\partial}{\partial X_3^P}, \frac{\partial}{\partial X_4^P})_\mu \,,$$
$$\vec{B}_P \to \vec{B}_P \,,$$
$$\vec{E}_P \to -\vec{E}_P \,,$$
$$\vec{s} \to \vec{s} \,,$$
$$\vec{q} \to -\vec{q} \,. \qquad (\mu = 1, ..., 4) \tag{5.13}$$

Hence one sees that with the building blocks:

$$\frac{d}{d\tau}, \partial^P, U^P, F^P, S$$

no pseudotensors occur in (5.5), so that (5.5) is covariant under the parity transformation. [21] With (5.13) one sees that $\vec{r}_P, \vec{v}_P, \vec{E}_P, \vec{q}$ are polar vectors whereas $\vec{B}_P, \vec{s}$ are axial vectors.

Next I consider the time inversion (=time reversal transformation). This acts in the following way:

$$\tau \to -\tau \,,$$
$$X_\mu^P \to (\vec{r}_P^{\,\dagger}, -X_4^P)_\mu \,,$$
$$U_\mu^P \to (-\gamma_P \cdot \vec{v}_P^{\,\dagger}, i \cdot \gamma_P)_\mu \,,$$
$$\partial_\mu^P \to (\frac{\partial}{\partial X_1^P}, \frac{\partial}{\partial X_2^P}, \frac{\partial}{\partial X_3^P}, -\frac{\partial}{\partial X_4^P})_\mu \,,$$
$$\vec{B}_P \to -\vec{B}_P \,,$$
$$\vec{E}_P \to \vec{E}_P \,,$$
$$\vec{s} \to -\vec{s} \,,$$
$$\vec{q} \to \vec{q} \,. \qquad (\mu = 1, ..., 4) \tag{5.14}$$

Now I use the fact that (5.12) is equivalent to (5.5). In fact, the application of (5.14) to (5.12b-c) shows first of all that the spatial parts of (5.5a-c) are covariant under the time reversal transformation. It then follows by using (3.8) to get

$$\dot{U}_4^P = -\frac{1}{U_4^P} \cdot (U_1^P \cdot \dot{U}_1^P + U_2^P \cdot \dot{U}_2^P + U_3^P \cdot \dot{U}_3^P) \,,$$
$$\dot{\vec{q}} = \dot{\vec{s}} \wedge \vec{v}_P + \frac{e}{m} \cdot \vec{s} \wedge (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P) - \frac{e}{m} \cdot \vec{v}_P^{\,\dagger} \cdot \vec{E}_P \cdot \vec{q} \,, \tag{5.15}$$

that the temporal parts of (5.5a-c) are covariant under the time reversal transformation. Hence one concludes that (5.5) is covariant under the time reversal transformation. In summary:

---

[21]Note that the direct product or the contraction of two tensors is again a tensor, not a pseudotensor. For the distinction between tensors and pseudotensors, see for example [Moe72].

- Equations (5.5) are covariant under the Poincaré group. Specifically, they are invariant under Poincaré transformations (except that the electromagnetic field transforms as described in section 3).

- As a special case the Frenkel equations (3.12) are Poincaré covariant from which it follows that also (2.11) is Poincaré covariant [q].

Because F^P transforms as a tensor field of rank 2 one also finds that (3.11) is invariant under the Poincaré group.

5.4

So far I have assumed that S^P is a tensor. Now for completeness and as announced in section 3 I demonstrate that the tensor property of S^P is consistent with the transformation properties of $\vec{v}_P$ under the Poincaré group, i.e. the relation $\vec{q} = \vec{s} \wedge \vec{v}_P$ is conserved under Poincaré transformations. In fact, by (5.13-14) one sees that it is conserved under spatial rotations, space inversion and time reversal and in the remaining part of this subsection I show that it is also conserved under the proper Lorentz transformation.

The infinitesimal proper Lorentz transformation (=infinitesimal Lorentz boost) is defined by

$$
\begin{aligned}
L_{jk}^{boost} &\equiv \delta_{jk} \ , \\
L_{j4}^{boost} &\equiv i \cdot v_{boost,j} \ , \\
L_{4j}^{boost} &\equiv -L_{j4}^{boost} \ , \\
L_{44}^{boost} &\equiv 1 \ , \qquad (j,k = 1,2,3)
\end{aligned}
\tag{5.16}
$$

where the infinitesimal vector $\vec{v}_{boost}$ denotes the relative velocity of the frames connected by L^{boost}. From (5.16) one obtains

$$
L_{\mu\nu}^{boost} \cdot L_{\mu\rho}^{boost} = \delta_{\nu\rho} \ , \qquad (\nu,\rho = 1,...,4)
\tag{5.17}
$$

which is consistent with the fact that L^{boost} belongs to the Lorentz group. The spin tensor transforms under L^{boost} via:

$$
S_{\mu\nu}^P \rightarrow S_{\mu\nu}^{P,boost} \equiv L_{\mu\rho}^{boost} \cdot L_{\nu\omega}^{boost} \cdot S_{\rho\omega}^P \ . \qquad (\mu,\nu = 1,...,4)
\tag{5.18}
$$

Abbreviating:

$$
S^{P,boost} \leftrightarrow \left(\vec{s}_{boost}, -i \cdot \vec{q}_{boost} \right) \ ,
$$

one sees by (5.18) that $\vec{s}, \vec{q}$ transform under L^{boost} via:

$$
\begin{aligned}
\vec{s} &\rightarrow \vec{s}_{boost} = \vec{s} - \vec{v}_{boost} \wedge \vec{q} \ , \\
\vec{q} &\rightarrow \vec{q}_{boost} = \vec{q} + \vec{v}_{boost} \wedge \vec{s} \ .
\end{aligned}
\tag{5.19}
$$

Because U^P is a 4-vector, one observes that $\vec{v}_P, \gamma_P$ transform under L^{boost} via:

$$
\begin{aligned}
\gamma_P &\rightarrow \gamma_{P,boost} \equiv \gamma_P \cdot \left(1 - \vec{v}_{boost}^\dagger \cdot \vec{v}_P \right) \ , \\
\vec{v}_P &\rightarrow \vec{v}_{P,boost} \equiv \vec{v}_P + \vec{v}_{boost}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P - \vec{v}_{boost} \ .
\end{aligned}
\tag{5.20}
$$

Collecting (5.19-20) one observes:

$$\vec{q}_{boost} = \vec{s}_{boost} \wedge \vec{v}_{P,boost} \; ,$$

so that the relation: $\vec{q} = \vec{s} \wedge \vec{v}_P$ is conserved under the infinitesimal proper Lorentz transformation L^{boost}. This concludes the proof that this relation is conserved under the whole Poincaré group.

5.5

Subsection 5.3 listed the transformation properties of $\vec{r}_P, t, \vec{v}_P, \vec{s}, \vec{B}_P, \vec{E}_P$ under the Poincaré group. Combining these with (2.1),(2.3a), (2.4-5),(2.7) one also obtains the transformation properties of $\vec{r}_M, t, \vec{v}_M, \vec{\sigma}, \vec{B}_M, \vec{E}_M$. One observes:

- The relations (2.1),(2.3a),(2.4-5),(2.7) are invariant under Poincaré transformations.

- By (2.7) the 'P' fields depend on $\vec{r}_P, t$ in the same way as the 'M' fields depend on $\vec{r}_M, t$.

Because (5.12) is Poincaré covariant, one concludes that also the DK equations (1.5) are Poincaré covariant. Specifically, they are invariant under Poincaré transformations (except that the electromagnetic 'M' fields transform in the same way as the 'P' fields) [r].

Note that $(\vec{r}_M^\dagger, i \cdot t)_\mu$ does not transform as a space-time position and $(\gamma_M \cdot \vec{v}_M^\dagger, i \cdot \gamma_M)_\mu$ is not a 4-vector but both transform *nonlinearly* in a complicated way. Analogously one obtains the well known property that the rest frame spin vector $\vec{\sigma}$ is not the spatial part of a 4-vector. Using (2.5) and the tensor property of S^P I now discuss the transformation properties of $\vec{\sigma}$. Firstly by (5.13), $\vec{\sigma}$ is, like $\vec{s}$, an axial vector and by (5.14) it transforms in the same way under time reversal.

Secondly, under the infinitesimal proper Lorentz transformation (5.16) it transforms via:

$$\vec{\sigma} \rightarrow \vec{\sigma} + \frac{\gamma_P}{\gamma_P + 1} \cdot \left(\vec{v}_{boost} \wedge \vec{v}_P \right) \wedge \vec{\sigma} \; , \tag{5.21}$$

which is simply a rotation associated with a change of orientation of the reference frame due to the boost. Thus $\vec{\sigma}$ has in fact the transformation properties of the rest frame spin vector [BMT59, Jac75, Tho27].

5.6

As already mentioned, $\vec{r}_M, t$ are the position and time variables used in the everyday business of accelerator physics. In fact via (2.1) and by neglecting spin: $\vec{r}_M = \vec{r}_P$, so that in this approximation

$$(\vec{r}_M^\dagger, i \cdot t)_\mu \qquad (\mu = 1, ..., 4) \tag{5.22}$$

transforms as a space-time position. Also one observes in this approximation that:

$$(\gamma_M \cdot \vec{v}_M^\dagger, i \cdot \gamma_M)_\mu \qquad (\mu = 1, ..., 4) \tag{5.23}$$

is a 4-vector. Even if one includes the spin one obtains these transformation properties if the SG-force is neglected. In fact in that approximation one gets:

$$\vec{r}_M' = \vec{v}_M, \tag{5.24a}$$

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = e \cdot (\vec{v}_M \wedge \vec{B}_M) + e \cdot \vec{E}_M \; , \tag{5.24b}$$

$$\vec{\sigma}\,' = \vec{\Omega}_M \wedge \vec{\sigma} \; . \tag{5.24c}$$

Note that (5.24) is studied in [BMT59, Jac75, Tho27].

6   Using the spin pseudo-4-vector

6.1

So far I have described the spin in terms of the spin tensor S^P but because the particle only has a intrinsic *magnetic* dipole moment and no intrinsic electric dipole moment one can also describe the spin just in terms of a pseudo-4-vector T^P defined by: [22]

$$T_\mu^P = -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot S_{\nu\rho}^P \cdot U_\omega^P , \qquad (\mu = 1, ..., 4) \tag{6.1}$$

where $\varepsilon_{\mu\nu\rho\omega}$ is the Levi-Civita symbol. [23] From this it follows that

$$S_{\mu\nu}^P = -i \cdot \varepsilon_{\mu\nu\rho\omega} \cdot T_\rho^P \cdot U_\omega^P . \qquad (\mu, \nu = 1, ..., 4) \tag{6.2}$$

The constraints (3.8) now read as:

$$U_\mu^P \cdot U_\mu^P = -1 ,$$
$$T_\mu^P \cdot U_\mu^P = 0 . \tag{6.3}$$

From (2.4-5),(3.2),(3.6) and (6.1) one has:

$$T_\mu^P = (\vec{T}_P^\dagger, T_4^P)_\mu , \qquad (\mu = 1, ..., 4)$$
$$\vec{T}_P = \frac{1}{\gamma_P} \cdot \vec{s} + \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P = \vec{\sigma} + \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} \cdot \vec{v}_P ,$$
$$T_4^P = i \cdot \vec{v}_P^\dagger \cdot \vec{T}_P = i \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{s} = i \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} . \tag{6.4}$$

Thus under the space inversion one gets by (5.13),(6.1):

$$T_\mu^P \to (\vec{T}_P^\dagger, -T_4^P)_\mu , \qquad (\mu = 1, ..., 4) \tag{6.5}$$

so that in fact T^P is a pseudo-4-vector. Under the time reversal one gets by (5.14),(6.1):

$$T_\mu^P \to (-\vec{T}_P^\dagger, T_4^P)_\mu . \qquad (\mu = 1, ..., 4) \tag{6.6}$$

6.2

Now I introduce the pseudotensor field $\tilde{F}^P$ dual to F^P defined by

$$\tilde{F}_{\mu\nu}^P = -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot F_{\rho\omega}^P ,$$
$$F_{\mu\nu}^P = \frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot \tilde{F}_{\rho\omega}^P , \qquad (\mu, \nu = 1, ..., 4) \tag{6.7}$$

---

[22]For more details on T^P, see for example [BMT59, Cor68, FG61a, Nyb64].
[23]Note that ε is the totally antisymmetric pseudotensor of rank 4 with $\varepsilon_{1234} = 1$.

from which it follows that:

$$\tilde{F}^P \leftrightarrow (-\vec{E}_P, -i \cdot \vec{B}_P) \ .$$

In Appendix E it is shown that (5.5) is equivalent to:

$$\dot{X}^P_\mu = U^P_\mu \ , \tag{6.8a}$$

$$\dot{U}^P_\mu = \frac{e}{m} \cdot F^P_{\mu\nu} \cdot U^P_\nu - \frac{e \cdot c_2}{2 \cdot m^2} \cdot U^P_\alpha \cdot T^P_\beta \cdot \partial^P_\mu \tilde{F}^P_{\alpha\beta} - \frac{e \cdot (c_1 + 2)}{2 \cdot m^2} \cdot U^P_\mu \cdot U^P_\alpha \cdot T^P_\beta \cdot U^P_\lambda \cdot \partial^P_\lambda \tilde{F}^P_{\alpha\beta}$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot T^P_\nu \cdot U^P_\lambda \cdot \partial^P_\lambda \tilde{F}^P_{\mu\nu} + \frac{e^2 \cdot (c_3 - 2 \cdot c_4 + c_5)}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot U^P_\nu \cdot U^P_\alpha \cdot T^P_\beta \cdot \tilde{F}^P_{\alpha\beta}$$

$$+ \frac{e^2 \cdot c_3}{4 \cdot m^3} \cdot F^P_{\mu\nu} \cdot \tilde{F}^P_{\nu\alpha} \cdot T^P_\alpha - \frac{e^2 \cdot c_5}{4 \cdot m^3} \cdot \tilde{F}^P_{\mu\nu} \cdot U^P_\nu \cdot U^P_\alpha \cdot T^P_\beta \cdot F^P_{\alpha\beta} \ , \tag{6.8b}$$

$$\dot{T}^P_\mu = \frac{e \cdot g}{2 \cdot m} \cdot F^P_{\mu\nu} \cdot T^P_\nu + \frac{e \cdot (g - 2)}{2 \cdot m} \cdot U^P_\mu \cdot T^P_\omega \cdot F^P_{\nu\omega} \cdot U^P_\nu \ . \qquad (\mu = 1, ..., 4) \tag{6.8c}$$

Note that (6.8) conserves the kinematic constraints (6.3) and (6.8c) is the BMT equation [BMT59]. Note also that (6.8) is covariant under the Poincaré group [24] because (5.5) is, too.

It follows from the normalization of the spin vector that:

$$T^P_\mu \cdot T^P_\mu = \hbar^2/4 \ . \tag{6.9}$$

As in (1.9) this equation is of second order in spin so that it plays no role in this paper. Note also that (6.9) is conserved under (6.8c).

7 The nonrelativistic limit. The rest frame

7.1

In this first subsection I consider the nonrelativistic limit (= zeroth order in $1/c$) and in the remainder of this section I consider the particle rest frame. I do this for the general case, i.e. for arbitrary values of $c_1, ..., c_5$. In the nonrelativistic limit (5.12) leads to: [25]

$$\vec{r}\,'_P = \vec{v}_P \ , \tag{7.1a}$$

$$m \cdot \vec{v}\,'_P = e \cdot \vec{E}_P + \frac{e \cdot c_2}{2 \cdot m} \cdot \vec{\nabla}_P (\vec{s}^\dagger \cdot \vec{B}_P) - \frac{e \cdot (c_2 - 2 - c_1)}{2 \cdot m} \cdot \left(\vec{s} \wedge \frac{\partial \vec{E}_P}{\partial t} \right)$$

$$+ \frac{e^2 \cdot c_3}{4 \cdot m^2} \cdot \vec{E}^\dagger_P \cdot \vec{B}_P \cdot \vec{s} + \frac{e^2}{4 \cdot m^2} \cdot (-c_3 + 2 \cdot c_4 - c_5) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P + \frac{e^2 \cdot c_5}{4 \cdot m^2} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P \ ,$$

$$\tag{7.1b}$$

$$\vec{s}\,' = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{s} \wedge \vec{B}_P) \ . \tag{7.1c}$$

One now sees that the second order SG terms survive even in the nonrelativistic limit. This should come as no surprise: second order terms are seen in the usual semi-relativistic Foldy-Wouthuysen transformations [b]. Note that the third term on the rhs of (7.1b) is sometimes

---

[24] i.e. (6.8) is invariant under Poincaré transformations (except that the electromagnetic field transforms in the prescribed way)

[25] The partial derivative $\partial/\partial t$ in (7.1), (7.2) and (7.5) acts on functions depending on $\vec{r}_P, t$.

called the 'magnetodynamic force'. In the nonrelativistic limit the Frenkel equations lead via (5.6),(7.1) to:

$$\vec{r}_P' = \vec{v}_P \ ,$$

$$m \cdot \vec{v}_P' = e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m} \cdot \vec{\nabla}_P (\vec{s}^{\dagger} \cdot \vec{B}_P) - \frac{e \cdot (g-2)}{2 \cdot m} \cdot (\vec{s} \wedge \frac{\partial \vec{E}_P}{\partial t})$$

$$+ \frac{e^2 \cdot (g-2) \cdot g}{4 \cdot m^2} \cdot \vec{E}_P^{\dagger} \cdot \vec{B}_P \cdot \vec{s} + \frac{e^2}{m^2} \cdot \vec{s}^{\dagger} \cdot \vec{B}_P \cdot \vec{E}_P - \frac{e^2 \cdot (g-2)^2}{4 \cdot m^2} \cdot \vec{s}^{\dagger} \cdot \vec{E}_P \cdot \vec{B}_P \ ,$$

$$\vec{s}\,' = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{s} \wedge \vec{B}_P) \ , \tag{7.2}$$

which is equivalent to: [26]

$$\vec{r}_M' = \vec{v}_M \ ,$$

$$m \cdot \vec{v}_M' = e \cdot \vec{E}_M + \frac{e \cdot g}{2 \cdot m} \cdot \vec{\nabla}_M (\vec{\sigma}^{\dagger} \cdot \vec{B}_M) - \frac{e \cdot g}{2 \cdot m} \cdot (\vec{\sigma} \wedge \frac{\partial \vec{E}_M}{\partial t})$$

$$+ \frac{e^2 \cdot (g+2) \cdot g}{4 \cdot m^2} \cdot \vec{E}_M^{\dagger} \cdot \vec{B}_M \cdot \vec{\sigma} - \frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \vec{\sigma}^{\dagger} \cdot \vec{E}_M \cdot \vec{B}_M \ ,$$

$$\vec{\sigma}\,' = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{\sigma} \wedge \vec{B}_M) \ , \tag{7.3}$$

In the nonrelativistic limit the GNR equations lead via (5.8),(7.1) to:

$$\vec{r}_P' = \vec{v}_P \ ,$$

$$m \cdot \vec{v}_P' = e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m} \cdot \vec{\nabla}_P (\vec{s}^{\dagger} \cdot \vec{B}_P) \ ,$$

$$\vec{s}\,' = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{s} \wedge \vec{B}_P) \ . \tag{7.4}$$

The choice (5.9) leads in the nonrelativistic limit via (7.1) to:

$$\vec{r}_P' = \vec{v}_P \ ,$$

$$m \cdot \vec{v}_P' = e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m} \cdot \vec{\nabla}_P (\vec{s}^{\dagger} \cdot \vec{B}_P) - \frac{e \cdot g}{2 \cdot m} \cdot (\vec{s} \wedge \frac{\partial \vec{E}_P}{\partial t})$$

$$+ \frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \vec{E}_P^{\dagger} \cdot \vec{B}_P \cdot \vec{s} - \frac{e^2 \cdot (g^2 - 2 \cdot g)}{4 \cdot m^2} \cdot \vec{s}^{\dagger} \cdot \vec{E}_P \cdot \vec{B}_P \ ,$$

$$\vec{s}\,' = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{s} \wedge \vec{B}_P) \ , \tag{7.5}$$

which agrees with [4.57] in [Cos94]. For the choice (5.11) one gets in the nonrelativistic limit via (7.1):

$$\vec{r}_P' = \vec{v}_P \ ,$$

$$m \cdot \vec{v}_P' = e \cdot \vec{E}_P \ ,$$

$$\vec{s}\,' = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{s} \wedge \vec{B}_P) \ . \tag{7.6}$$

To summarize the above one observes that the characteristic parameters $c_3, -c_3 + 2 \cdot c_4 - c_5, c_5$ only vanish if $0 = c_3 = c_4 = c_5$. Using simple linear algebra it follows from (7.1b) that

---

[26]The partial derivative $\partial / \partial t$ in (7.3) acts on functions depending on $\vec{r}_M, t$. Note also that in the nonrelativistic limit one has: $\vec{s} = \vec{\sigma}$.

knowledge of the nonrelativistic limit (7.1) uniquely determines the values of the characteristic parameters $c_1, ..., c_5$. As an immediate application I note that the particle described in [CPP95] (see eq. (1.1) thereof) has a nonrelativistic limit identical to (7.4). If this particle belongs to the class obeying (5.5), one can conclude that the particle described in [CPP95] obeys the GNR equations, i.e. the characteristic parameters $c_1, ..., c_5$ for this particle assume the values given by (5.8). Thus [CPP95] provides a good example of how to identify $c_1, ..., c_5$. That in fact the particle described in [CPP95] is characterized by the choice (5.8) of the parameters $c_1, ..., c_5$ is supported by the results of section 8.

7.2

Although the behaviour in the nonrelativistic limit is closely related to that in the rest frame, which I define as the frame for which the velocity $\vec{v}_P$ vanishes, the two behaviours are nevertheless distinct. In the nonrelativistic limit I am observing the motion in a selected inertial frame but motion with respect to the rest frame is actually motion with respect to an accelerated frame. The Lorentz transformation L^R to the particle rest frame transforms X^P to the rest frame space-time position X^R:

$$X^R_\mu = L^R_{\mu\nu} \cdot X^P_\nu , \qquad (\mu = 1, ..., 4) \tag{7.7}$$

where

$$L^R_{jk} = (\gamma_P - 1) \cdot v_{P,j} \cdot v_{P,k} \cdot \frac{1}{\vec{v}^\dagger_P \cdot \vec{v}_P} + \delta_{jk} ,$$

$$L^R_{j4} = i \cdot \gamma_P \cdot v_{P,j} ,$$

$$L^R_{4j} = -L^R_{j4} ,$$

$$L^R_{44} = \gamma_P . \qquad (j, k = 1, 2, 3) \tag{7.8}$$

Since the rest frame is an accelerated frame, L^R depends on the proper time. Relative to the rest frame the particle motion vanishes, i.e.

$$U^R_\mu \equiv L^R_{\mu\nu} \cdot U^P_\nu = (0, i)_\mu . \qquad (\mu = 1, ..., 4) \tag{7.9}$$

In the rest frame the quantities S^P, F^P transform to:

$$S^R_{\mu\nu} \equiv L^R_{\mu\rho} \cdot L^R_{\nu\omega} \cdot S^P_{\rho\omega} = \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 & 0 \\ -\sigma_3 & 0 & \sigma_1 & 0 \\ \sigma_2 & -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} , \tag{7.10a}$$

$$F^R_{\mu\nu}(X^R_1, ..., X^R_4) \equiv L^R_{\mu\rho} \cdot L^R_{\nu\omega} \cdot F^P_{\rho\omega}(X^P_1, ..., X^P_4) , \qquad (\mu, \nu = 1, ..., 4) \tag{7.10b}$$

where I also used (3.5). Note that on combining (7.8),(7.10a) one obtains (3.6). Furthermore one concludes from (6.4),(7.8):

$$T^R_\mu = L^R_{\mu\nu} \cdot T^P_\nu = (\vec{\sigma}^\dagger, 0)_\mu , \qquad (\mu = 1, ..., 4) \tag{7.11}$$

where T^R denotes the spin pseudo-4-vector w.r.t. the rest frame.

If $N_\mu^R(\tau)$ transforms under the Lorentz group as a 4-vector or as a space–time position and $N_{\mu\nu}^R(\tau)$ as a tensor of rank 2, the rest frame proper time derivative $(d/d\tau)_R$ is defined by

$$(\frac{d}{d\tau})_R N_\mu^R = L_{\mu\nu}^R \cdot \frac{d}{d\tau}(L_{\nu\rho}^{R\ -1} \cdot N_\rho^R) ,$$

$$(\frac{d}{d\tau})_R N_{\mu\nu}^R = L_{\mu\lambda}^R \cdot L_{\nu\rho}^R \cdot \frac{d}{d\tau}(L_{\lambda\alpha}^{R\ -1} \cdot L_{\rho\beta}^{R\ -1} \cdot N_{\alpha\beta}^R) . \qquad (\mu, \nu = 1, ..., 4) \qquad (7.12)$$

The distinction between $(d/d\tau)_R$ and $d/d\tau$ takes into account the proper time dependence of the rest frame which occurs because in general the 4-velocity U^P is not a constant of motion. Applying (7.12) to the quantities X^R, U^R, S^R yields:

$$(\frac{d}{d\tau})_R X_\mu^R = L_{\mu\nu}^R \cdot \dot{X}_\nu^P ,$$

$$(\frac{d}{d\tau})_R U_\mu^R = L_{\mu\nu}^R \cdot \dot{U}_\nu^P ,$$

$$(\frac{d}{d\tau})_R S_{\mu\nu}^R = L_{\mu\rho}^R \cdot L_{\nu\omega}^R \cdot \dot{S}_{\rho\omega}^P . \qquad (\mu, \nu = 1, ..., 4) \qquad (7.13)$$

Combining (5.1) with (7.13) results in

$$(\frac{d}{d\tau})_R X_\mu^R = U_\mu^R , \qquad (7.14a)$$

$$(\frac{d}{d\tau})_R U_\mu^R = \frac{e}{m} \cdot L_{\mu\nu}^R \cdot F_{\nu\rho}^P \cdot U_\rho^P + L_{\mu\nu}^R \cdot Y_\nu^P , \qquad (7.14b)$$

$$(\frac{d}{d\tau})_R S_{\mu\nu}^R = L_{\mu\alpha}^R \cdot L_{\nu\beta}^R \cdot \left(\frac{e \cdot g}{2 \cdot m} \cdot [F_{\alpha\omega}^P \cdot S_{\omega\beta}^P - S_{\alpha\omega}^P \cdot F_{\omega\beta}^P] \right.$$
$$\left. - \frac{e \cdot (g-2)}{2 \cdot m} \cdot [S_{\alpha\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\beta^P - S_{\beta\omega}^P \cdot F_{\omega\lambda}^P \cdot U_\lambda^P \cdot U_\alpha^P] \right), \quad (\mu, \nu = 1, ..., 4) \quad (7.14c)$$

where in (7.14a) I also used (7.9). To simplify (7.14) I introduce the abbreviation

$$\partial_\mu^R \equiv (\frac{\partial}{\partial X_1^R} , \frac{\partial}{\partial X_2^R} , \frac{\partial}{\partial X_3^R} , \frac{\partial}{\partial X_4^R})_\mu , \qquad (\mu = 1, ..., 4) \qquad (7.15)$$

from which it follows by (7.10b) that: [27]

$$\partial_\mu^P F_{\nu\rho}^P = L_{\lambda\mu}^R \cdot L_{\alpha\nu}^R \cdot L_{\beta\rho}^R \cdot \partial_\lambda^R F_{\alpha\beta}^R . \qquad (\mu, \nu, \rho = 1, ..., 4) \qquad (7.16)$$

Combining (5.4),(7.10),(7.16) one gets

$$L_{\mu\nu}^R \cdot Y_\nu^P = -\frac{e \cdot c_2}{4 \cdot m^2} \cdot \left(S_{\nu\omega}^R \cdot \partial_\mu^R F_{\omega\nu}^R + U_\mu^R \cdot S_{\nu\omega}^R \cdot U_\lambda^R \cdot \partial_\lambda^R F_{\omega\nu}^R \right)$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot S_{\mu\nu}^R \cdot U_\omega^R \cdot U_\lambda^R \cdot \partial_\lambda^R F_{\nu\omega}^R + \frac{e^2 \cdot c_3}{4 \cdot m^3} \cdot F_{\mu\nu}^R \cdot S_{\nu\omega}^R \cdot F_{\omega\rho}^R \cdot U_\rho^R$$

$$- \frac{e^2 \cdot c_4}{4 \cdot m^3} \cdot F_{\mu\nu}^R \cdot U_\nu^R \cdot S_{\omega\rho}^R \cdot F_{\rho\omega}^R + \frac{e^2 \cdot c_5}{4 \cdot m^3} \cdot S_{\mu\nu}^R \cdot F_{\nu\rho}^R \cdot F_{\rho\omega}^R \cdot U_\omega^R , \qquad (\mu = 1, ..., 4) \quad (7.17)$$

---

[27]The partial derivatives ∂_μ^R always act on functions depending on $X_1^R, X_2^R, X_3^R, X_4^R$.

so that (7.14) specifies to:

$$\left(\frac{d}{d\tau}\right)_R X_\mu^R = U_\mu^R \, , \tag{7.18a}$$

$$\left(\frac{d}{d\tau}\right)_R U_\mu^R = \frac{e}{m} \cdot F_{\mu\nu}^R \cdot U_\nu^R - \frac{e \cdot c_2}{4 \cdot m^2} \cdot \left(S_{\nu\omega}^R \cdot \partial_\mu^R F_{\omega\nu}^R + U_\mu^R \cdot S_{\nu\omega}^R \cdot U_\lambda^R \cdot \partial_\lambda^R F_{\omega\nu}^R \right)$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot S_{\mu\nu}^R \cdot U_\omega^R \cdot U_\lambda^R \cdot \partial_\lambda^R F_{\nu\omega}^R + \frac{e^2 \cdot c_3}{4 \cdot m^3} \cdot F_{\mu\nu}^R \cdot S_{\nu\omega}^R \cdot F_{\omega\rho}^R \cdot U_\rho^R$$

$$- \frac{e^2 \cdot c_4}{4 \cdot m^3} \cdot F_{\mu\nu}^R \cdot U_\nu^R \cdot S_{\omega\rho}^R \cdot F_{\rho\omega}^R + \frac{e^2 \cdot c_5}{4 \cdot m^3} \cdot S_{\mu\nu}^R \cdot F_{\nu\rho}^R \cdot F_{\rho\omega}^R \cdot U_\omega^R \, , \tag{7.18b}$$

$$\left(\frac{d}{d\tau}\right)_R S_{\mu\nu}^R = \frac{e \cdot g}{2 \cdot m} \cdot \left(F_{\mu\omega}^R \cdot S_{\omega\nu}^R - S_{\mu\omega}^R \cdot F_{\omega\nu}^R \right)$$

$$- \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \left(S_{\mu\omega}^R \cdot F_{\omega\lambda}^R \cdot U_\lambda^R \cdot U_\nu^R - S_{\nu\omega}^R \cdot F_{\omega\lambda}^R \cdot U_\lambda^R \cdot U_\mu^R \right) \, . \quad (\mu, \nu = 1, ..., 4) \tag{7.18c}$$

Note that (7.18) has the same form as (5.5). Thus the use of $(d/d\tau)_R$ provides an economic formulation of the rest frame equations of motion. To compare the rest frame equations (7.18) with the nonrelativistic behaviour studied in the previous subsection I introduce the abbreviations

$$X_\mu^R = (\vec{r}_R^{\,\dagger}, X_4^R)_\mu \, , \qquad U_\mu^R = (\gamma_R \cdot \vec{v}_R^{\,\dagger}, i \cdot \gamma_R)_\mu \, ,$$
$$F^R \leftrightarrow (\vec{B}_R, -i \cdot \vec{E}_R) \, . \qquad (\mu = 1, ..., 4) \tag{7.19}$$

The simple structures of U^R and S^R allow the spatial parts of equations (7.18a-c) to be easily obtained and one gets: [28]

$$\left(\frac{d}{d\tau}\right)_R \vec{r}_R = 0 \, , \tag{7.20a}$$

$$m \cdot \left(\frac{d}{d\tau}\right)_R \vec{v}_R = e \cdot \vec{E}_R + \frac{e \cdot c_2}{2 \cdot m} \cdot \vec{\nabla}_R (\vec{\sigma}^\dagger \cdot \vec{B}_R) - \frac{i \cdot e \cdot (c_2 - 2 - c_1)}{2 \cdot m} \cdot \left(\vec{\sigma} \wedge \frac{\partial \vec{E}_R}{\partial X_4^R} \right)$$

$$+ \frac{e^2 \cdot c_3}{4 \cdot m^2} \cdot \vec{E}_R^\dagger \cdot \vec{B}_R \cdot \vec{\sigma} + \frac{e^2}{4 \cdot m^2} \cdot (-c_3 + 2 \cdot c_4 - c_5) \cdot \vec{\sigma}^\dagger \cdot \vec{B}_R \cdot \vec{E}_R + \frac{e^2 \cdot c_5}{4 \cdot m^2} \cdot \vec{\sigma}^\dagger \cdot \vec{E}_R \cdot \vec{B}_R \, , \tag{7.20b}$$

$$\left(\frac{d}{d\tau}\right)_R \vec{\sigma} = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{\sigma} \wedge \vec{B}_R) \, . \tag{7.20c}$$

By (7.20a) one sees that the rest frame is that frame where the derivative $(d/d\tau)_R$ of the position vanishes. Note that (7.20) also displays the fact that the rest frame is an *accelerated* frame. In fact, by the definition of $(d/d\tau)_R$ one sees that the rest frame proper time derivative of a vanishing quantity in general is nonvanishing, which is exemplified by (7.20a-b):

$$\left(\frac{d}{d\tau}\right)_R \left(\frac{d}{d\tau}\right)_R \vec{r}_R = \left(\frac{d}{d\tau}\right)_R \vec{v}_R \neq 0 \, .$$

7.3

In the special case of the Frenkel equations it is interesting to consider the rest frame behaviour of Π^P instead of U^P. Abbreviating:

$$\Pi_\mu^R \equiv L_{\mu\nu}^R \cdot \Pi_\nu^P \equiv (\vec{\pi}_R^{\,\dagger}, \Pi_4^R)_\mu \, , \qquad (\mu = 1, ..., 4) \tag{7.21}$$

---

[28] The nabla operator $\vec{\nabla}_R$ acts on functions depending on $\vec{r}_R, -i \cdot X_4^R, \vec{\sigma}$ and it is the gradient w.r.t. $\vec{r}_R$.

one obtains by (4.9),(7.12):

$$(\frac{d}{d\tau})_{R}\Pi_{\mu}^{R} = \frac{e}{m} \cdot F_{\mu\nu}^{R} \cdot U_{\nu}^{R} - \frac{e \cdot g}{4 \cdot m^{2}} \cdot S_{\nu\omega}^{R} \cdot \partial_{\mu}^{R} F_{\omega\nu}^{R} \ . \qquad (\mu = 1,...,4) \qquad (7.22)$$

Together with (7.20a),(7.20c) one then gets:

$$(\frac{d}{d\tau})_{R}\vec{r}_{R} = 0 \ , \qquad\qquad\qquad (7.23a)$$

$$(\frac{d}{d\tau})_{R}\vec{\pi}_{R} = e \cdot \vec{E}_{R} + \frac{e \cdot g}{2 \cdot m} \cdot \vec{\nabla}_{R}(\vec{\sigma}^{\dagger} \cdot \vec{B}_{R}) \ , \qquad\qquad (7.23b)$$

$$(\frac{d}{d\tau})_{R}\vec{\sigma} = \frac{e \cdot g}{2 \cdot m} \cdot (\vec{\sigma} \wedge \vec{B}_{R}) \ . \qquad\qquad (7.23c)$$

Comparing with (7.20) one sees that in the rest frame the canonical momentum vector for the Frenkel equations fulfills the same equation of motion as the kinetic momentum vector of the GNR equations. Note that (7.23) was obtained also in [Pla66a].

8 Estimating the strength of the SG force in magnets

8.1

This paper is partly motivated by the suggestion [CPP95, NR87] that the SG force can be used to separate spin states, either in real space or 'energy space', in (anti-)proton storage rings. Now that I have general forms for the relativistic SG force I am in a position to carry this study further. In this section I will apply my equations of motion to the HERA proton ring (HERA-p) [Br95]. I do this for the general case, i.e. for arbitrary values of $c_1, ..., c_5$. Note that for (anti-)protons one has: $(g - 2)/2 \approx 1.79$.

8.2

In this subsection I study the equations of motion of the Pryce coordinates. I will only treat the case of static (i.e. time independent) magnetic fields and vanishing electric fields, i.e.: [29]

$$\frac{\partial \vec{B}_{P}}{\partial t} = 0 \ , \qquad \vec{E}_{P} = 0 \ . \qquad\qquad (8.1)$$

I leave it to the reader to investigate other cases. To facilitate the numerical evaluations I will use Gaussian units [30] so that I drop the convention: $c = 1$. In the case of a static magnetic field (5.12b) reduces to:

$$m \cdot (\gamma_{P} \cdot \vec{v}_{P})' = \frac{e}{c} \cdot (\vec{v}_{P} \wedge \vec{B}_{P}) + \frac{e \cdot c_{2}}{2 \cdot m \cdot c \cdot \gamma_{P}} \cdot \vec{\nabla}_{P}(\vec{s}^{\dagger} \cdot \vec{B}_{P})$$

$$+ \frac{e \cdot \gamma_{P}}{2 \cdot m \cdot c^{3}} \cdot [(c_{1} + 2) \cdot \vec{s}^{\dagger} \cdot \vec{B}_{P}' \cdot \vec{v}_{P} + (c_{2} - c_{1} - 2) \cdot \vec{v}_{P}^{\dagger} \cdot \vec{s} \cdot \vec{B}_{P}']$$

$$+ \frac{e^{2}}{4 \cdot m^{2} \cdot c^{4}} \cdot [c_{5} \cdot \vec{v}_{P}^{\dagger} \cdot \vec{B}_{P} \cdot (\vec{B}_{P} \wedge \vec{s}) - c_{5} \cdot \vec{B}_{P}^{\dagger} \cdot \vec{B}_{P} \cdot (\vec{v}_{P} \wedge \vec{s})$$

$$+ (2 \cdot c_{4} - c_{3}) \cdot \vec{s}^{\dagger} \cdot \vec{B}_{P} \cdot (\vec{v}_{P} \wedge \vec{B}_{P})] \ . \qquad\qquad (8.2)$$

---

[29]The partial derivative $\partial/\partial t$ in (8.1) acts on functions depending on $\vec{r}_{P}, t$.

[30]See for example [Jac75].

To simplify the comparison with [CPP95], I express $\vec{s}$ in terms of $\vec{\sigma}$ in (8.2). This leads by (2.4) to: [31]

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = \frac{e}{c} \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+ \frac{e \cdot c_2}{2 \cdot m \cdot c \cdot \gamma_P} \cdot \hat{\vec{\nabla}}_P \left(\vec{B}_P^\dagger \cdot [\gamma_P \cdot \vec{\sigma} - \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m \cdot c^3} \cdot [(c_1 + 2) \cdot (\gamma_P \cdot \vec{\sigma} - \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P)^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$+ (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} \cdot \vec{B}_P'] + \frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot [c_5 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{B}_P \wedge (\gamma_P \cdot \vec{\sigma}$$

$$- \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P) - c_5 \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{\sigma})$$

$$+ (2 \cdot c_4 - c_3) \cdot \vec{B}_P^\dagger \cdot (\gamma_P \cdot \vec{\sigma} - \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P) \cdot (\vec{v}_P \wedge \vec{B}_P)]$$

$$= \frac{e}{c} \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m \cdot c \cdot \gamma_P} \cdot \hat{\vec{\nabla}}_P \left(\gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{\sigma} - \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m \cdot c^3} \cdot [(c_1 + 2) \cdot (\gamma_P \cdot \vec{\sigma} - \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P)^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$+ (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} \cdot \vec{B}_P'] + \frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot [c_5 \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{\sigma})$$

$$+ \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot (-c_5 + 2 \cdot c_4 - c_3) \cdot \vec{\sigma}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{v}_P)$$

$$- c_5 \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{\sigma}) + (2 \cdot c_4 - c_3) \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{\sigma} \cdot (\vec{v}_P \wedge \vec{B}_P)]$$

$$= \frac{e}{c} \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \hat{\vec{\nabla}}_P (\vec{\sigma}^\dagger \cdot \vec{B}_P) + \frac{e}{2 \cdot m \cdot c^3} \cdot [\gamma_P^2 \cdot (c_1 + 2) \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$- (c_1 + 2) \cdot \frac{\gamma_P^3}{c^2 \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$+ \gamma_P \cdot (c_2 \cdot \frac{\gamma_P}{\gamma_P + 1} - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} \cdot \vec{B}_P'] + \frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot [c_5 \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{\sigma})$$

$$+ \frac{\gamma_P^2}{c^2 \cdot (\gamma_P + 1)} \cdot (-c_5 + 2 \cdot c_4 - c_3) \cdot \vec{v}_P^\dagger \cdot \vec{\sigma} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{v}_P)$$

$$- c_5 \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{\sigma}) + (2 \cdot c_4 - c_3) \cdot \gamma_P \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)] . \quad (8.3)$$

I say that the magnetic field is 'transverse' if [32]

$$\vec{v}_P^\dagger \cdot \vec{B}_P = 0 ,$$
$$\vec{B}_P' = 0 . \quad (8.4)$$

Hence for a static, transverse magnetic field (8.3) simplifies to:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = \frac{e}{c} \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \hat{\vec{\nabla}}_P (\vec{\sigma}^\dagger \cdot \vec{B}_P)$$

[31] The nabla operator $\hat{\vec{\nabla}}_P$ in this paper always acts on functions depending on $\vec{r}_P, t, \vec{v}_P, \vec{\sigma}$ and it is the gradient w.r.t. $\vec{r}_P$.

[32] This corresponds, for example, to the case of a particle travelling instantaneously parallel to the axis in a pure quadrupole magnetic field or to a particle travelling instantaneously perpendicular to a pure dipole field.

$$+\frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot [-c_5 \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{\sigma}) + (2 \cdot c_4 - c_3) \cdot \gamma_P \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)] \ . \ (8.5)$$

If the magnetic field is transverse and the spin is parallel to the magnetic field then (8.5) simplifies to:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = \frac{e}{c} \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \hat{\vec{\nabla}}_P (\vec{\sigma}^\dagger \cdot \vec{B}_P)$$

$$+\frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot \gamma_P \cdot (-c_5 + 2 \cdot c_4 - c_3) \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) \ . \tag{8.6}$$

Thus with $c_2 = g$ and when using Pryce coordinates the first order SG force equals the SG force given in [CPP95]. Note that the SG force in (8.6) does not depend on c_1.

For protons one has

$$\frac{e \cdot \hbar}{m \cdot c} \ = \ 10^{-23} \cdot ergs \cdot Gauss^{-1} = 6.3 \cdot 10^{-18} \cdot MeV \cdot Gauss^{-1} \ . \tag{8.7}$$

In the HERA proton ring at about 800 GeV the magnetic field B_{HD} in the bending magnets is about 45 kGauss and the quadrupole magnetic field gradient G_{HQ} is about 9 kGauss/cm. Thus one has

$$\frac{e^2 \cdot B_{HD}^2 \cdot \hbar}{m^2 \cdot c^3} = 6.5 \cdot 10^{-21} \cdot dyne = 4.1 \cdot 10^{-15} \cdot MeV \cdot cm^{-1} \ ,$$

$$\frac{e \cdot G_{HQ} \cdot \hbar}{m \cdot c} = 9.1 \cdot 10^{-20} \cdot dyne = 5.7 \cdot 10^{-14} \cdot MeV \cdot cm^{-1} \ . \tag{8.8}$$

For the static, transverse magnetic field with spin vector $\vec{\sigma}$ parallel to the magnetic field the numerical values (8.8) lead via (8.6) to the following maximal values of the SG force:

$$\frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \hat{\vec{\nabla}}_P (\vec{\sigma}^\dagger \cdot \vec{B}_P) \ \rightarrow \ c_2 \cdot 2.3 \cdot 10^{-20} \cdot dyne = c_2 \cdot 1.4 \cdot 10^{-14} \cdot MeV \cdot cm^{-1} \ ,$$

$$\frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot (-c_5 + 2 \cdot c_4 - c_3) \cdot \gamma_P \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) \ \rightarrow$$

$$(-c_5 + 2 \cdot c_4 - c_3) \cdot \gamma_P \cdot 8.2 \cdot 10^{-22} \cdot dyne = (-c_5 + 2 \cdot c_4 - c_3) \cdot \gamma_P \cdot 5.1 \cdot 10^{-16} \cdot MeV \cdot cm^{-1}$$

$$= (-c_5 + 2 \cdot c_4 - c_3) \cdot 7.3 \cdot 10^{-19} \cdot dyne = (-c_5 + 2 \cdot c_4 - c_3) \cdot 4.6 \cdot 10^{-13} \cdot MeV \cdot cm^{-1} \ ,$$

$$\tag{8.9}$$

where I assumed $\gamma_P = 900$.

Thus at HERA-p energies and for the Frenkel equations (where one has: $-c_5 + 2 \cdot c_4 - c_3 = 4$) (see (5.6)), the second order SG force in the dipoles completely outweighs the first order SG force in the quadrupoles. It is also simple to show that the second order force for a particle travelling about 1 mm off axis through a quadrupole is negligible compared to both of the above. In the case of the GNR force (see (5.8)) the second order force is zero. Clearly, if the SG force is to be utilized in high energy proton storage rings, one must first decide which equation of motion appertains.

I say that the magnetic field is 'longitudinal' if

$$\vec{v}_P \wedge \vec{B}_P = 0 \ ,$$

$$\vec{B}_P' = \frac{1}{\vec{v}_P^\dagger \cdot \vec{v}_P} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P \ , \tag{8.10}$$

so that for a static, longitudinal magnetic field one gets:

$$\vec{B}_P = \frac{1}{\vec{v}_P^{\dagger} \cdot \vec{v}_P} \cdot \vec{v}_P^{\dagger} \cdot \vec{B}_P \cdot \vec{v}_P \ .$$

Hence for a static, longitudinal magnetic field (8.3) simplifies to:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = \frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \frac{1}{\vec{v}_P^{\dagger} \cdot \vec{v}_P} \cdot \gamma_P \cdot \vec{v}_P^{\dagger} \cdot \vec{\sigma} \cdot \vec{B}_P' \ . \tag{8.11}$$

Note that the second order SG terms have disappeared and that the force only depends on c_2. If the spin is parallel to the field this also agrees with the form given in [CPP95] if $c_2 = g$. However, because c_1 does not appear in (8.6) and (8.11), one sees that for these special magnetic fields, the GNR and reduced Frenkel equations are identical.

I consider a gradient G_{long} of the longitudinal field along the longitudinal direction of about 9 kGauss/cm[CPP95], so that:

$$\frac{e \cdot G_{long} \cdot \hbar}{m \cdot c} = 9.1 \cdot 10^{-20} \cdot dyne = 5.7 \cdot 10^{-14} \cdot MeV \cdot cm^{-1} \ . \tag{8.12}$$

Thus for the static, longitudinal magnetic field with spin vector $\vec{\sigma}$ parallel to the magnetic field the numerical value (8.12) leads via (8.11) to the following maximal value of the SG force:

$$\frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \frac{1}{\vec{v}_P^{\dagger} \cdot \vec{v}_P} \cdot \gamma_P \cdot \vec{v}_P^{\dagger} \cdot \vec{\sigma} \cdot \vec{B}_P' \ \rightarrow \ c_2 \cdot \gamma_P \cdot 2.3 \cdot 10^{-20} \cdot dyne$$
$$= c_2 \cdot \gamma_P \cdot 1.4 \cdot 10^{-14} \cdot MeV \cdot cm^{-1} = c_2 \cdot 2 \cdot 10^{-17} \cdot dyne = c_2 \cdot 1.3 \cdot 10^{-11} \cdot MeV \cdot cm^{-1} \ . \tag{8.13}$$

This agrees with [CPP95] where it is pointed out that the longitudinal SG force is much larger than the first order transverse force. It is also much larger than the second order transverse force. However, before this force could be used to separate the spin ensemble into two parts [CPP95], a way must be found to overcome the severe mixing [Hof96] caused by incoherent synchrotron oscillations. For a survey of other problems see [Der95, Der90b].

Because for the GNR equations $c_2 = g$, the identity of (8.6),(8.11) with the corresponding results in [CPP95] is consistent with the supposition in section 7 that the particle described in [CPP95] obeys the GNR equations, i.e. belongs to the choice (5.8) of the characteristic parameters $c_1, ..., c_5$.

8.3

This paper opened with a description of the DK Hamiltonian. This is based on the 'M' variables and so far I have only used these to provide the missing link connecting the DK equations to special relativity. However, since many other investigations [BHR94a, BHR94b, Der90a, Der90b] have been based on this Hamiltonian it is natural that one inspects the equations of motion for the 'M' variables in more detail. As in the previous subsection I will only treat the case of static (i.e. time independent) magnetic fields and vanishing electric fields, i.e.: [33]

$$\frac{\partial \vec{B}_M}{\partial t} = 0 \ , \qquad \vec{E}_M = 0 \ . \tag{8.14}$$

---

[33]The partial derivative $\partial / \partial t$ in (8.14) acts on functions depending on $\vec{r}_M, t$.

Note that by (2.7) the conditions (8.1),(8.14) are equivalent.

I begin by using Appendix C to rewrite (8.3) as:

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = \frac{e}{c} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{2 \cdot m \cdot c} \cdot (c_2 - 2 + \frac{2}{\gamma_M}) \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$+ \frac{e}{2 \cdot m \cdot c^3} \cdot \Big([(c_2 - g) \cdot \frac{\gamma_M^2}{\gamma_M + 1} - c_1 \cdot \gamma_M] \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}'_M$$

$$+ (c_1 \cdot \gamma_M^2 + 2 \cdot \gamma_M) \cdot \vec{\sigma}^\dagger \cdot \vec{B}'_M \cdot \vec{v}_M$$

$$+ \frac{1}{c^2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot [\frac{\gamma_M}{\gamma_M + 1} \cdot (g - 2) - c_1 \cdot \gamma_M] \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}'_M \cdot \vec{v}_M$$

$$+ (g - 2) \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}'_M \cdot \vec{\sigma} \Big)$$

$$+ \frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot \Big([4 - 4 \cdot \gamma_M + \gamma_M \cdot (2 \cdot c_4 - c_3)] \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$+ \frac{1}{c^2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot [-2 \cdot c_4 + c_3 + c_5 + 4 + (g^2 - 2 \cdot g) \cdot \frac{1}{\gamma_M + 1}] \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$+ [(g - 2) \cdot g \cdot \frac{1}{\gamma_M + 1} - \gamma_M \cdot c_5] \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{\sigma} \wedge \vec{B}_M) - c_5 \cdot \gamma_M \cdot \vec{B}_M^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{\sigma})$$

$$+ (g - 2)^2 \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{B}_M$$

$$- \frac{1}{c^2} \cdot (g^2 - 2 \cdot g) \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \Big) . \quad (8.15)$$

If the magnetic field is transverse and the spin is parallel to the magnetic field then (8.15) simplifies to:

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = \frac{e}{c} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{2 \cdot m \cdot c} \cdot (c_2 - 2 + \frac{2}{\gamma_M}) \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$+ \frac{e^2}{4 \cdot m^2 \cdot c^4} \cdot [4 - 4 \cdot \gamma_M + \gamma_M \cdot (2 \cdot c_4 - c_3 - c_5)] \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M) , \quad (8.16)$$

which for the Frenkel equations (see (5.6)) becomes:

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = \frac{e}{c} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{2 \cdot m \cdot c} \cdot (g - 2 + \frac{2}{\gamma_M}) \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$+ \frac{e^2}{m^2 \cdot c^4} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M) . \quad (8.17)$$

For the GNR case (see (5.8)) equation (8.16) becomes:

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = \frac{e}{c} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{2 \cdot m \cdot c} \cdot (g - 2 + \frac{2}{\gamma_M}) \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$- \frac{e^2}{m^2 \cdot c^4} \cdot (\gamma_M - 1) \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M) . \quad (8.18)$$

The first order parts of (8.17) and (8.18) are identical.

Although I have obtained the equations for the kinetic momentum, Hamiltonians lead more naturally to equations of motion for canonical momenta. If the magnetic field is transverse and

the spin is parallel to the magnetic field then by (B.4),(B.6),(B.17),(B.25-26) and by taking just the first order SG terms the Frenkel case gives:

$$
\begin{aligned}
\vec{\pi}'_M &= \frac{e}{c} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{2 \cdot m \cdot c} \cdot (g - 2 + \frac{2}{\gamma_M}) \cdot \vec{\nabla}_M(\vec{\sigma}^{\dagger} \cdot \vec{B}_M) \\
&\quad - \frac{e \cdot (g-2)}{2 \cdot m \cdot c^3} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}^{\dagger}_M \cdot \vec{\sigma} \cdot \vec{B}'_M ,
\end{aligned} \tag{8.19}
$$

and

$$
\begin{aligned}
\vec{p}'_M &= \frac{e}{c} \cdot (\vec{v}_M \wedge \vec{B}_M) - e \cdot \vec{\nabla}_M \phi_M + \frac{e}{c} \cdot (\vec{v}^{\dagger}_M \cdot \vec{\nabla}_M) \vec{A}_M \\
&\quad + \frac{e}{2 \cdot m \cdot c} \cdot (g - 2 + \frac{2}{\gamma_M}) \cdot \vec{\nabla}_M(\vec{\sigma}^{\dagger} \cdot \vec{B}_M) - \frac{e \cdot (g-2)}{2 \cdot m \cdot c^3} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}^{\dagger}_M \cdot \vec{\sigma} \cdot \vec{B}'_M .
\end{aligned} \tag{8.20}
$$

In equations (8.15-20) the first order SG piece contains the factor $(g/2 - 1 + 1/\gamma_M)$ which differs from the corresponding term for 'P' variables in (8.6) by the term $(-1 + 1/\gamma_M)$ [Hof95]. This term can be directly traced back to a similar term in $\vec{\Omega}_M$ in (1.6c) associated with Thomas precession [Jac76]. [34]

Further differences appear for the second order terms both for the Frenkel and GNR equations. Thus naive use of the DK Hamiltonian to obtain estimates of the relativistic SG force could lead one to the conclusion that it gives a first order SG force very different from that predicted by [CPP95]. In particular, for electrons one has: $(g - 2)/2 \approx 0.00116$ so that the Thomas term causes a massive relative change.

However, it is clear that the difference is only an artifact of the choice of position coordinates [DH95]: the rest frame implied by the 'M' variables is different from that of the 'P' variables and the corresponding Thomas precession terms are different. Furthermore, although the forces on $\vec{r}_M$ and $\vec{r}_P$ can be rather different, the position variable $\vec{r}_M$ is always closer to $\vec{r}_P$ than the particle Compton wave length (see (2.1)). Of course, the corresponding equations of motion conserve this property in time.

The equation of motion (2.36) given in [CJKP96] also contains the $(-1 + 1/\gamma_M)$ term and thereby appears to differ from [CPP95]. This should now come as no surprise since the former works with Newton-Wigner coordinates.

9 Estimating the strength of the SG force in electromagnetic traps

9.1

Most accounts of the SG force emphasize the first order component. But now that one has seen that the second order force can be important at high energy in large fields it is interesting to estimate their effect under other circumstances where the SG forces play a central role. An obvious case is that of the nonrelativistic SG force in electromagnetic traps [DSV86]. As in the previous section I use Gaussian units and I consider the general case, i.e. arbitrary values of $c_1, ..., c_5$. By (7.1b) one has: [35]

$$
m \cdot \vec{v}'_P = e \cdot \vec{E}_P + \frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \vec{\nabla}_P(\vec{\sigma}^{\dagger} \cdot \vec{B}_P) - \frac{e \cdot (c_2 - 2 - c_1)}{2 \cdot m \cdot c^2} \cdot (\vec{\sigma} \wedge \frac{\partial \vec{E}_P}{\partial t})
$$

---

[34]Note that in the nonrelativistic limit there is no difference (see (7.2-3) or put $\gamma_M = 1$) .

[35]The partial derivative $\partial/\partial t$ in (9.1) and (9.2) acts on functions depending on $\vec{r}_P, t$.

$$+\frac{e^2 \cdot c_3}{4 \cdot m^2 \cdot c^3} \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{\sigma} + \frac{e^2}{4 \cdot m^2 \cdot c^3} \cdot (-c_3 + 2 \cdot c_4 - c_5) \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$+\frac{e^2 \cdot c_5}{4 \cdot m^2 \cdot c^3} \cdot \vec{\sigma}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P \ . \tag{9.1}$$

If the magnetic field and electric field are parallel to the spin then (9.1) simplifies to:

$$m \cdot \vec{v}_P' = e \cdot \vec{E}_P + \frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \vec{\nabla}_P(\vec{\sigma}^\dagger \cdot \vec{B}_P) - \frac{e \cdot (c_2 - 2 - c_1)}{2 \cdot m \cdot c^2} \cdot (\vec{\sigma} \wedge \frac{\partial \vec{E}_P}{\partial t})$$

$$+\frac{e^2 \cdot c_4}{2 \cdot m^2 \cdot c^3} \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{\sigma} \ . \tag{9.2}$$

9.2

To apply my equations to the case of the electron I calculate

$$\frac{e \cdot \hbar}{m \cdot c} = 1.9 \cdot 10^{-20} \cdot ergs \cdot Gauss^{-1} = 1.16 \cdot 10^{-14} \cdot MeV \cdot Gauss^{-1} \ . \tag{9.3}$$

In one of the traps used in [DSV86] the magnetic field B_{trap} is about 20 kGauss and the electric field E_{trap} is about 0.000033 Statvolt/cm=1000 V/meter $\hat{=}$ 61 Gauss. Then I get

$$\frac{e^2 \cdot B_{trap} \cdot E_{trap} \cdot \hbar}{m^2 \cdot c^3} = 7.3 \cdot 10^{-21} \cdot dyne = 4.5 \cdot 10^{-15} \cdot MeV \cdot cm^{-1} \ . \tag{9.4}$$

If the electric and magnetic fields are static and parallel to the spin then the numerical value (9.4) leads via (9.2) to the following maximal value of the second order SG force

$$\frac{e^2 \cdot c_4}{2 \cdot m^2 \cdot c^3} \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{\sigma} \rightarrow c_4 \cdot 1.8 \cdot 10^{-21} \cdot dyne = c_4 \cdot 1.1 \cdot 10^{-15} \cdot MeV \cdot cm^{-1} \ . \tag{9.5}$$

The magnetic field has a gradient G_{trap} of about 2.4 Gauss/cm so that

$$\frac{e \cdot G_{trap} \cdot \hbar}{m \cdot c} = 4.5 \cdot 10^{-20} \cdot dyne = 2.8 \cdot 10^{-14} \cdot MeV \cdot cm^{-1} \ . \tag{9.6}$$

For the static, transverse magnetic field when the spin vector $\vec{\sigma}$ is parallel to the magnetic field the numerical value (9.6) leads via (9.1) to the following maximal values of the first order SG force:

$$\frac{e \cdot c_2}{2 \cdot m \cdot c} \cdot \hat{\vec{\nabla}}_P(\vec{\sigma}^\dagger \cdot \vec{B}_P) \rightarrow c_2 \cdot 1.1 \cdot 10^{-20} \cdot dyne = c_2 \cdot 6.9 \cdot 10^{-15} \cdot MeV \cdot cm^{-1} \ . \tag{9.7}$$

Thus for the Frenkel case ((5.6) with $g \approx 2$) the second order SG force is comparable to, but still smaller than, the first order force. For the GNR case ((5.8) with $g \approx 2$) the first order force is unchanged but the second order force vanishes.

For protons the values corresponding to (9.5),(9.7) are smaller and the second order SG force is much smaller than the first order force.

My main purpose in presenting these numbers is to compare the different terms of the SG force. I do not claim that they give a good representation of the real SG force in a trap because my formalism only applies to the semiclassical regime and this may not always be directly applicable to traps.

10 A transformation of the GNR equations

10.1

In this section I deal with transformations which in particular allow the GNR equations to be transformed into the reduced Frenkel equations. This would, for example, make it possible to solve the GNR equations by symplectic methods.

10.2

The general equations (5.5) in the approximation that the second order SG terms are neglected read as:

$$\dot{X}^P_\mu = U^P_\mu \,, \tag{10.1a}$$

$$\dot{U}^P_\mu = \frac{e}{m} \cdot F^P_{\mu\nu} \cdot U^P_\nu - \frac{e \cdot c_2}{4 \cdot m^2} \cdot \left(S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} + U^P_\mu \cdot S^P_{\nu\omega} \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\omega\nu} \right)$$

$$+ \frac{e \cdot (c_2 - c_1 - 2)}{2 \cdot m^2} \cdot S^P_{\mu\nu} \cdot U^P_\omega \cdot U^P_\lambda \cdot \partial^P_\lambda U^P_{\nu\omega} \,, \tag{10.1b}$$

$$\dot{S}^P_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \left(F^P_{\mu\omega} \cdot S^P_{\omega\nu} - S^P_{\mu\omega} \cdot F^P_{\omega\nu} \right)$$

$$- \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \left(S^P_{\mu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\nu - S^P_{\nu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\mu \right) \,. \qquad (\mu, \nu = 1, ..., 4) \tag{10.1c}$$

Equations (10.1) are obtained by setting $c_3 = c_4 = c_5 = 0$ in (5.5). In the case where $c_2 = g$ equations (10.1) simplify to:

$$\dot{X}^P_\mu = U^P_\mu \,, \tag{10.2a}$$

$$\dot{U}^P_\mu = \frac{e}{m} \cdot F^P_{\mu\nu} \cdot U^P_\nu - \frac{e \cdot g}{4 \cdot m^2} \cdot \left(S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} + U^P_\mu \cdot S^P_{\nu\omega} \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\omega\nu} \right)$$

$$+ \frac{e \cdot (g - 2 - c_1)}{2 \cdot m^2} \cdot S^P_{\mu\nu} \cdot U^P_\omega \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\nu\omega} \,, \tag{10.2b}$$

$$\dot{S}^P_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \left(F^P_{\mu\omega} \cdot S^P_{\omega\nu} - S^P_{\mu\omega} \cdot F^P_{\omega\nu} \right)$$

$$- \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \left(S^P_{\mu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\nu - S^P_{\nu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\mu \right) \,. \quad (\mu, \nu = 1, ..., 4) \tag{10.2c}$$

For $c_1 = g - 2$, equations (10.2) are the GNR equations and for $c_1 = 0$ they are the reduced Frenkel equations.

10.3

I now demonstrate how to relate equations (10.2) corresponding to different values of c_1. I do this by defining a transformation of X^P and S^P via the rule:

$$X^P_\mu \longrightarrow X_\mu \,, \qquad S^P_{\mu\nu} \longrightarrow S_{\mu\nu} = S^P_{\mu\nu} \,, \qquad (\mu, \nu = 1, ..., 4) \tag{10.3a}$$

with

$$U_\mu \equiv \dot{X}_\mu \equiv U_\mu^P + \frac{e \cdot \Delta c_1}{2 \cdot m^2} \cdot S_{\mu\nu}^P \cdot F_{\nu\omega}^P \cdot U_\omega^P \, , \qquad (\mu = 1, ..., 4) \qquad (10.3b)$$

where Δc_1 is a real number. One sees that the spin tensor does not change under (10.3), so that this transformation only has an effect on the SG force, as seen below. The transformation (10.3) is a straightforward generalization of a transformation given in [Pla66a]. The constraints (3.8) are equivalent to:

$$U_\mu \cdot U_\mu = -1 \, , \qquad\qquad\qquad (10.4a)$$
$$S_{\mu\nu} \cdot U_\nu = 0 \, . \qquad (\mu = 1, ..., 4) \qquad\qquad (10.4b)$$

On introducing the abbreviations [36]

$$X_\mu = (\vec{r}^{\,\dagger}, X_4)_\mu \, ,$$
$$\partial_\mu \equiv (\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}, \frac{\partial}{\partial X_4})_\mu \, , \qquad (\mu = 1, ..., 4) \qquad (10.5)$$

I define: [37]

$$\vec{E} \equiv \vec{E}_M(\vec{r}, -i \cdot X_4) \, , \qquad \vec{B} \equiv \vec{B}_M(\vec{r}, -i \cdot X_4) \, ,$$
$$F \leftrightarrow (\vec{B}, -i \cdot \vec{E}) \, . \qquad\qquad\qquad (10.6)$$

Now I come to the main conclusion of this section: if X^P, U^P, S^P obey (10.2), then X, U, S obey

$$\dot{X}_\mu = U_\mu \, , \qquad\qquad\qquad\qquad (10.7a)$$
$$\dot{U}_\mu = \frac{e}{m} \cdot F_{\mu\nu} \cdot U_\nu - \frac{e \cdot g}{4 \cdot m^2} \cdot \left(S_{\nu\omega} \cdot \partial_\mu F_{\omega\nu} + U_\mu \cdot S_{\nu\omega} \cdot U_\lambda \cdot \partial_\lambda F_{\omega\nu} \right)$$
$$+ \frac{e \cdot (g - 2 - c_1 + \Delta c_1)}{2 \cdot m^2} \cdot S_{\mu\nu} \cdot U_\omega \cdot U_\lambda \cdot \partial_\lambda F_{\nu\omega} \, , \qquad (10.7b)$$
$$\dot{S}_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \left(F_{\mu\omega} \cdot S_{\omega\nu} - S_{\mu\omega} \cdot F_{\omega\nu} \right)$$
$$- \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \left(S_{\mu\omega} \cdot F_{\omega\lambda} \cdot U_\lambda \cdot U_\nu - S_{\nu\omega} \cdot F_{\omega\lambda} \cdot U_\lambda \cdot U_\mu \right) \, , \quad (\mu, \nu = 1, ..., 4) \qquad (10.7c)$$

where all second order SG terms are neglected in (10.7b). In the special case where $c_1 = g - 2$ equations (10.7) read as:

$$\dot{X}_\mu = U_\mu \, , \qquad\qquad\qquad\qquad (10.8a)$$
$$\dot{U}_\mu = \frac{e}{m} \cdot F_{\mu\nu} \cdot U_\nu - \frac{e \cdot g}{4 \cdot m^2} \cdot \left(S_{\nu\omega} \cdot \partial_\mu F_{\omega\nu} + U_\mu \cdot S_{\nu\omega} \cdot U_\lambda \cdot \partial_\lambda F_{\omega\nu} \right)$$

---

[36] The partial derivatives ∂_μ always act on functions depending on X_1, X_2, X_3, X_4.

[37] My notation is chosen so as to indicate that the functional dependence of $\vec{E}$ on $\vec{r}, -i \cdot X_4$ is the same as the dependence of $\vec{E}_M$ on $\vec{r}_M, t$ and likewise for $\vec{B}_M$. Note that F depends on X_1, X_2, X_3, X_4 in the same way as F^P depends on $X_1^P, X_2^P, X_3^P, X_4^P$.

$$+\frac{e \cdot \Delta c_1}{2 \cdot m^2} \cdot S_{\mu\nu} \cdot U_\omega \cdot U_\lambda \cdot \partial_\lambda F_{\nu\omega} \ , \tag{10.8b}$$

$$\dot{S}_{\mu\nu} = \frac{e \cdot g}{2 \cdot m} \cdot \left(F_{\mu\omega} \cdot S_{\omega\nu} - S_{\mu\omega} \cdot F_{\omega\nu} \right)$$

$$-\frac{e \cdot (g-2)}{2 \cdot m} \cdot \left(S_{\mu\omega} \cdot F_{\omega\lambda} \cdot U_\lambda \cdot U_\nu - S_{\nu\omega} \cdot F_{\omega\lambda} \cdot U_\lambda \cdot U_\mu \right) \ , \quad (\mu, \nu = 1, ..., 4) \tag{10.8c}$$

where all second order SG terms are neglected in (10.8b). Therefore the GNR equations are transformed under (10.3) into equations (10.8). In particular, with the choice: $\Delta c_1 = g - 2$, one has transformed (10.8) into the reduced Frenkel equations. Thus one has transformed the GNR equations into the reduced Frenkel equations. Therefore the GNR equations can be solved by solving the DK equations and inverting (10.3b) so that one can use symplectic methods [BHR]. For practical applications in accelerator physics it is helpful that (10.1-8) contain the charge e only up to first order.

It follows from the normalization of the spin vector that:

$$S_{\mu\nu} \cdot S_{\mu\nu} = \hbar^2/2 \ . \tag{10.9}$$

As in (1.9) this equation is of second order in spin so that it plays no role in this paper. Note also that (10.9) is conserved under (10.7c).

Summary

I have studied classical spin-orbit systems at first order in spin and, by applying dimensional analysis and imposing Poincaré covariance, I have found that these spin-orbit systems are characterized by five dimensionless parameters $c_1, ..., c_5$. My axiomatic approach is supported by the observation that the most prominent spin-orbit systems, namely those of Frenkel and GNR, are special cases of my scheme.

In this approach, i.e. at first order in spin, the five parameters are purely phenomenological and are to be determined by experiment. For example the Frenkel and GNR equations give very different SG forces at high energy in proton storage rings. There are also differences for high fields in traps. Theory is of little help. For example, as I pointed out in [o] even the Dirac equation cannot deliver unambiguous answers. In the three cases mentioned the parameters all depend on g. However, protons, for example, have substructure and one should not assume a priori that the dependence of the $c's$ on the $g's$ is the same for all particles.

In this paper I have concentrated on spin 1/2 particles. Nevertheless my results are formulated classically so that they can be applied to particles of arbitrary spin.

In addition I have devoted special attention to the DK equations and have found a transformation of the particle coordinates which relates these equations to the Frenkel equations. The DK equations are therefore (nonmanifestly) Poincaré covariant. The new coordinates differ from the original coordinates by less than the Compton wave length and the corresponding time variables are the same. Thus one concludes that the particles described by both equations are effectively indistinguishable.

As I have just pointed out, different values of the $c's$, correspond to different systems of spin-orbit equations and can lead to very different SG forces. Thus, before proposing techniques which rely on SG forces at high energy, one must decide which equations are applicable. Alternatively one can take the view that a measurement of the forces is in itself a way of discovering which equations to use.

Appendix A

A.1

In this Appendix I derive (2.9) from section 1 and subsection 2.1. I introduce the abbreviation:

$$\vec{\Omega}_P \equiv -\frac{e}{m} \cdot \left([\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P - \frac{g-2}{2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \right.$$
$$\left. -[\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P) \right), \tag{A.1}$$

and conclude from (2.4),(A.1):

$$\vec{s}\,' = \left(\gamma_M \cdot \vec{\sigma} - \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M \right)' = \gamma_M' \cdot \vec{\sigma} + \gamma_M \cdot \vec{\sigma}\,' - \left(\frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M \right)'$$

$$= \frac{e}{m} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{\sigma} + \gamma_M \cdot (\vec{\Omega}_M \wedge \vec{\sigma}) + \frac{e}{m} \cdot \frac{\gamma_M^2}{(\gamma_M + 1)^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M$$

$$- \frac{\gamma_M}{\gamma_M + 1} \cdot \left(\gamma_M \cdot \vec{v}_M^\dagger \cdot (\vec{\Omega}_M \wedge \vec{\sigma}) \cdot \vec{v}_M + \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot (\gamma_M \cdot \vec{v}_M)' + \vec{\sigma}^\dagger \cdot (\gamma_M \cdot \vec{v}_M)' \cdot \vec{v}_M \right)$$

$$= \frac{e}{m} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot [\frac{1}{\gamma_M} \cdot \vec{s} + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M] + \gamma_M \cdot \vec{\Omega}_M \wedge [\frac{1}{\gamma_M} \cdot \vec{s} + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M]$$

$$+ \frac{e}{m} \cdot \frac{\gamma_M^2}{(\gamma_M + 1)^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M$$

$$- \frac{\gamma_M^2}{\gamma_M + 1} \cdot (\vec{v}_M \wedge \vec{\Omega}_M)^\dagger \cdot [\frac{1}{\gamma_M} \cdot \vec{s} + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M] \cdot \vec{v}_M$$

$$- \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot [\vec{v}_M \wedge \vec{B}_M + \vec{E}_M] - \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot [\vec{v}_M \wedge \vec{B}_M + \vec{E}_M] \cdot \vec{v}_M$$

$$= \frac{e}{m} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] + \gamma_P \cdot \vec{\Omega}_P \wedge [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]$$

$$+ \frac{e}{m} \cdot \frac{\gamma_P^2}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P$$

$$+ \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{v}_P \wedge \vec{s})^\dagger \cdot \vec{\Omega}_P \cdot \vec{v}_P - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P]$$

$$- \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot \vec{v}_P - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{E}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s}$$

$$+ \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \cdot \vec{v}_P$$

$$= \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s} + \frac{e}{m} \cdot \frac{2 \cdot \gamma_P^2 + \gamma_P}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P$$

$$+ \frac{e}{m} \cdot [\vec{s} + \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \wedge \left([\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P - \frac{g-2}{2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \right.$$

$$\left. -[\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P) \right)$$

$$+ \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \left([\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P - \frac{g-2}{2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \right.$$

$$\left. -[\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P) \right) \cdot \vec{v}_P - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P]$$

$$
\begin{aligned}
&-\frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot \vec{v}_P - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{E}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} \\
&\quad + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \cdot \vec{v}_P \\
=\ & \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s} + \frac{e}{m} \cdot \frac{2 \cdot \gamma_P^2 + \gamma_P}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \\
&\quad + \frac{e}{m} \cdot [\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot (\vec{s} \wedge \vec{B}_P) + \frac{e}{m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot [\frac{g-2}{2} + \frac{1}{\gamma_P}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) \\
&+ \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) - \frac{e}{m} \cdot [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot [\vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P - \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P] \\
&\quad - \frac{e}{m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot [\vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P - \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \\
&\quad + \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot [\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{v}_P \\
&\quad - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot [\vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P - \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{E}_P] \cdot \vec{v}_P \\
&\quad - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P) \\
&\quad - \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot \vec{v}_P - \frac{e}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{E}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} \\
&\quad + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \cdot \vec{v}_P \\
=\ & \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s} - \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \\
&\quad + \frac{e}{m} \cdot [\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot (\vec{s} \wedge \vec{B}_P) + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) \\
&+ \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) - \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P \\
&\quad + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{v}_P\ .
\end{aligned} \tag{A.2}
$$

Introducing the abbreviations [38]

$$
\begin{aligned}
\vec{\Delta}_0 \equiv\ & \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s} - \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \\
&+ \frac{e}{m} \cdot [\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot (\vec{s} \wedge \vec{B}_P) + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) \\
&+ \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) - \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P \\
&+ \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{v}_P\ , \\
\vec{\Delta}_1 \equiv\ & \frac{e}{m} \cdot [\frac{g-2}{2} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P + \vec{v}_P \wedge \vec{B}_P) + \frac{1}{\gamma_P} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}
\end{aligned}
$$

---
[38]Note that all $\Delta's$ are first order in spin.

$$-\frac{g-2}{2}\cdot\gamma_P\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\vec{v}_P+\frac{g}{2}\cdot\frac{1}{\gamma_P}\cdot(\vec{s}\wedge\vec{B}_P)-\frac{g}{2}\cdot\frac{1}{\gamma_P}\cdot\vec{s}^\dagger\cdot\vec{E}_P\cdot\vec{v}_P]\ ,$$

$$\vec{\Delta}_2\equiv\vec{\Delta}_1-\vec{\Delta}_0=\frac{e}{2\cdot m}\cdot\left((g-2)\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot(\vec{v}_P\wedge\vec{B}_P)\right.$$

$$+(g-2)\cdot\frac{1-\gamma_P}{\gamma_P}\cdot(\vec{s}\wedge\vec{B}_P)-(g-2)\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot(\vec{v}_P\wedge\vec{s})$$

$$\left.-(g-2)\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{B}_P^\dagger\cdot(\vec{s}\wedge\vec{v}_P)\cdot\vec{v}_P\right)\ , \tag{A.3}$$

one sees by (A.2) that (2.9) is valid if $\vec{\Delta}_2$ vanishes. Hence the remaining task of Appendix A is to show that $\vec{\Delta}_2$ vanishes.

A.2

For the case where $\vec{v}_P,\vec{B}_P$ are linearly independent (e.g. nonparallel), one has the following 3 linearly independent vectors:

$$\vec{v}_P,\vec{B}_P,\vec{v}_P\wedge\vec{B}_P\ .$$

One concludes from (A.3):

$$\vec{v}_P^\dagger\cdot\vec{\Delta}_2=\frac{e\cdot(g-2)}{2\cdot m}\cdot\vec{v}_P^\dagger\cdot(\vec{s}\wedge\vec{B}_P)\cdot[\frac{1-\gamma_P}{\gamma_P}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{v}_P]=0\ ,$$

$$\vec{B}_P^\dagger\cdot\vec{\Delta}_2=0\ ,$$

$$(\vec{v}_P\wedge\vec{B}_P)^\dagger\cdot\vec{\Delta}_2=\frac{e}{2\cdot m}\cdot(\vec{v}_P\wedge\vec{B}_P)^\dagger\cdot\left((g-2)\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot(\vec{v}_P\wedge\vec{B}_P)\right.$$

$$+(g-2)\cdot\frac{1-\gamma_P}{\gamma_P}\cdot(\vec{s}\wedge\vec{B}_P)-(g-2)\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot(\vec{v}_P\wedge\vec{s})$$

$$\left.-(g-2)\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{B}_P^\dagger\cdot(\vec{s}\wedge\vec{v}_P)\cdot\vec{v}_P\right)$$

$$=\ \frac{e\cdot(g-2)}{2\cdot m}\cdot\left(\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot[\vec{v}_P^\dagger\cdot\vec{v}_P\cdot\vec{B}_P^\dagger\cdot\vec{B}_P-\vec{B}_P^\dagger\cdot\vec{v}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_P]\right.$$

$$+\frac{1-\gamma_P}{\gamma_P}\cdot[\vec{v}_P^\dagger\cdot\vec{s}\cdot\vec{B}_P^\dagger\cdot\vec{B}_P-\vec{s}^\dagger\cdot\vec{B}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_P]$$

$$\left.-\frac{\gamma_P}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot[\vec{v}_P^\dagger\cdot\vec{v}_P\cdot\vec{B}_P^\dagger\cdot\vec{s}-\vec{v}_P^\dagger\cdot\vec{s}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P]\right)=0\ . \tag{A.4}$$

Because $\vec{v}_P,\vec{B}_P,\vec{v}_P\wedge\vec{B}_P$ constitute a basis of vectors, one concludes by (A.4) that $\vec{\Delta}_2$ vanishes for the case where $\vec{v}_P,\vec{B}_P$ are linearly independent.

To discuss the case where $\vec{v}_P,\vec{B}_P$ are linearly dependent, one first observes by (A.3) that $\vec{\Delta}_2$ vanishes, if $\vec{v}_P=0$ or $\vec{B}_P=0$. It remains to consider the subcase with: $\vec{B}_P=\lambda\cdot\vec{v}_P$, where λ is a constant which balances the dimensions. Then from (A.3) follows

$$\vec{\Delta}_2=\frac{e\cdot\lambda\cdot(g-2)}{2\cdot m}\cdot\left(\frac{1-\gamma_P}{\gamma_P}\cdot(\vec{s}\wedge\vec{v}_P)-\frac{\gamma_P}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{v}_P\cdot(\vec{v}_P\wedge\vec{s})\right)=0\ . \tag{A.5}$$

Hence I have shown that $\vec{\Delta}_2$ vanishes. This completes the proof of (2.9).

Appendix B

B.1

In this Appendix I derive (2.10) from subsections 1.1-2 and subsections 2.1-2. This task is tedious but straightforward. In fact I only have to consider $m \cdot (\gamma_P \cdot \vec{v}_P)'$ as determined by the Hamiltonian H_M through the relation: [39]

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = \{m \cdot \gamma_P \cdot \vec{v}_P, H_M\}_M + \frac{\partial}{\partial t}(m \cdot \gamma_P \cdot \vec{v}_P) \ .$$

Hence $m \cdot (\gamma_P \cdot \vec{v}_P)'$ is a well defined function of $\vec{r}_M, t, \vec{p}_M, \vec{\sigma}$ and the main task is to reexpress it as a function of $\vec{r}_P, t, \vec{v}_P, \vec{s}$.

B.2

First of all I express $\vec{v}'_M$ in terms of $\vec{r}_M, t, \vec{v}_M, \vec{\sigma}$ and I abbreviate: [40]

$$K_M \equiv m \cdot \gamma_M \ ,$$
$$\vec{\Delta}_3 \equiv \{\vec{r}_M, \vec{\sigma}^\dagger \cdot \vec{W}_M\}_M = \check{\vec{\nabla}}_M(\vec{\sigma}^\dagger \cdot \vec{W}_M) \ . \tag{B.1}$$

To simplify (B.1) I calculate for an arbitrary function $f(\vec{\pi}_M)$:

$$\check{\vec{\nabla}}_M(f \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M) = \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \check{\vec{\nabla}}_M f \ ,$$
$$\check{\vec{\nabla}}_M(f \cdot \vec{\sigma}^\dagger \cdot \vec{\pi}_M) = \vec{\sigma}^\dagger \cdot \vec{\pi}_M \cdot \check{\vec{\nabla}}_M f + f \cdot \vec{\sigma} \ ,$$
$$\check{\vec{\nabla}}_M(f \cdot \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M)) = \check{\vec{\nabla}}_M(f \cdot \vec{\pi}_M^\dagger \cdot (\vec{E}_M \wedge \vec{\sigma})) = \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \check{\vec{\nabla}}_M f + f \cdot (\vec{E}_M \wedge \vec{\sigma}) \ ,$$
$$\check{\vec{\nabla}}_M J_M = \frac{\vec{\pi}_M}{J_M} \ , \qquad \check{\vec{\nabla}}_M \frac{1}{J_M} = -\frac{\vec{\pi}_M}{J_M^3} \ , \tag{B.2}$$

from which follows by (B.1):

$$\begin{aligned}
\vec{\Delta}_3 &= \check{\vec{\nabla}}_M(\vec{\sigma}^\dagger \cdot \vec{W}_M) = -\frac{e}{m} \cdot \left(\vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \check{\vec{\nabla}}_M[\frac{m}{J_M} + \frac{g-2}{2}] \right. \\
&\quad -\frac{g-2}{2} \cdot [\vec{\sigma}^\dagger \cdot \vec{\pi}_M \cdot \check{\vec{\nabla}}_M(\frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M) + \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma}] \\
&\quad \left. -\vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \check{\vec{\nabla}}_M[\frac{g}{2 \cdot J_M} - \frac{1}{J_M + m}] - [\frac{g}{2 \cdot J_M} - \frac{1}{J_M + m}] \cdot (\vec{E}_M \wedge \vec{\sigma}) \right) \\
&= -\frac{e}{m} \cdot \left(-\frac{m}{J_M^3} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M - \frac{g-2}{2} \cdot \vec{\sigma}^\dagger \cdot \vec{\pi}_M \cdot [\frac{1}{J_M \cdot (J_M + m)} \cdot \vec{B}_M \right. \\
&\quad -\frac{m + 2 \cdot J_M}{J_M^3 \cdot (J_M + m)^2} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M] - \frac{g-2}{2} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \\
&\quad \left. +[\frac{g}{2 \cdot J_M^3} - \frac{1}{J_M \cdot (J_M + m)^2}] \cdot \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \vec{\pi}_M - [\frac{g}{2 \cdot J_M} - \frac{1}{J_M + m}] \cdot (\vec{E}_M \wedge \vec{\sigma}) \right) \\
&= \frac{e}{J_M^3} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M + \frac{e}{m \cdot J_M \cdot (J_M + m)^2} \cdot \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \vec{\pi}_M
\end{aligned}$$

---

[39] Here the partial derivative $\partial/\partial t$ acts on functions depending on $\vec{r}_M, t, \vec{p}_M, \vec{\sigma}$.

[40] The nabla operator $\check{\vec{\nabla}}_M$ always acts on functions depending on $\vec{r}_M, t, \vec{p}_M, \vec{\sigma}$ and it is the gradient w.r.t. $\vec{p}_M$.

$$-\frac{e}{m \cdot (J_M + m)} \cdot (\vec{E}_M \wedge \vec{\sigma}) + \frac{e \cdot g}{2 \cdot m \cdot J_M} \cdot (\vec{E}_M \wedge \vec{\sigma}) - \frac{e \cdot g}{2 \cdot m \cdot J_M^3} \cdot \vec{\sigma}^\dagger \cdot (\vec{\pi}_M \wedge \vec{E}_M) \cdot \vec{\pi}_M$$

$$+\frac{g-2}{2} \cdot \Big(\frac{e}{m} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\sigma}^\dagger \cdot \vec{\pi}_M \cdot \vec{B}_M$$

$$-\frac{e}{m} \cdot \frac{m + 2 \cdot J_M}{J_M^3 \cdot (J_M + m)^2} \cdot \vec{\sigma}^\dagger \cdot \vec{\pi}_M \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M + \frac{e}{m} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \Big)$$

$$= \frac{e}{K_M^2} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + \frac{e \cdot K_M}{m \cdot (K_M + m)^2} \cdot \vec{\sigma}^\dagger \cdot (\vec{v}_M \wedge \vec{E}_M) \cdot \vec{v}_M - \frac{e}{m \cdot (K_M + m)} \cdot (\vec{E}_M \wedge \vec{\sigma})$$

$$+\frac{e \cdot g}{2 \cdot m \cdot K_M} \cdot (\vec{E}_M \wedge \vec{\sigma}) - \frac{e \cdot g}{2 \cdot m \cdot K_M} \cdot \vec{\sigma}^\dagger \cdot (\vec{v}_M \wedge \vec{E}_M) \cdot \vec{v}_M$$

$$+\frac{g-2}{2} \cdot \Big(\frac{e}{m} \cdot \frac{1}{K_M + m} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M - \frac{e}{m} \cdot \frac{m + 2 \cdot K_M}{(K_M + m)^2} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M$$

$$+\frac{e}{m} \cdot \frac{1}{K_M + m} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \Big) . \tag{B.3}$$

Also one gets from (1.4),(B.1-2):

$$\frac{\vec{\pi}_M}{J_M} = \breve{\vec{\nabla}}_M J_M = \{\vec{r}_M, J_M\}_M = \{\vec{r}_M, H_M\}_M - \{\vec{r}_M, \vec{\sigma}^\dagger \cdot \vec{W}_M\}_M = \vec{v}_M - \vec{\Delta}_3 . \tag{B.4}$$

Next I abbreviate

$$\vec{\Delta}_4 \equiv \{\vec{\pi}_M, \vec{\sigma}^\dagger \cdot \vec{W}_M\}_M ,$$
$$\vec{\Delta}_5 \equiv \vec{\Delta}_3' ,$$
$$\Delta_6 \equiv m \cdot \gamma_M^3 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3 , \tag{B.5}$$

from which follows by (1.3-4): [41]

$$\vec{\pi}_M' = \{\vec{\pi}_M, H_M\}_M + \frac{\partial \vec{\pi}_M}{\partial t} = \frac{e}{J_M} \cdot (\vec{\pi}_M \wedge \vec{B}_M) + e \cdot \vec{E}_M + \{\vec{\pi}_M, \vec{\sigma}^\dagger \cdot \vec{W}_M\}_M$$

$$= \frac{e}{J_M} \cdot (\vec{\pi}_M \wedge \vec{B}_M) + e \cdot \vec{E}_M + \vec{\Delta}_4 ,$$

$$J_M \cdot J_M' = \vec{\pi}_M^\dagger \cdot \vec{\pi}_M' = e \cdot \vec{E}_M^\dagger \cdot \vec{\pi}_M + \vec{\pi}_M^\dagger \cdot \vec{\Delta}_4 ,$$

$$J_M' = \frac{e}{J_M} \cdot \vec{E}_M^\dagger \cdot \vec{\pi}_M + \frac{1}{J_M} \cdot \vec{\pi}_M^\dagger \cdot \vec{\Delta}_4 = \frac{e}{J_M} \cdot \vec{E}_M^\dagger \cdot \vec{\pi}_M + \vec{v}_M^\dagger \cdot \vec{\Delta}_4 , \tag{B.6}$$

so that one gets from (B.1),(B.4):

$$\vec{v}_M' = \frac{\vec{\pi}_M'}{J_M} - \frac{J_M'}{J_M^2} \cdot \vec{\pi}_M + \vec{\Delta}_3' = \frac{e}{J_M^2} \cdot (\vec{\pi}_M \wedge \vec{B}_M) + \frac{e}{J_M} \cdot \vec{E}_M + \frac{1}{J_M} \cdot \vec{\Delta}_4 - \frac{e}{J_M^3} \cdot \vec{E}_M^\dagger \cdot \vec{\pi}_M \cdot \vec{\pi}_M$$

$$-\frac{1}{J_M^2} \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_4 \cdot \vec{\pi}_M + \vec{\Delta}_5 = \frac{e}{J_M^2} \cdot (\vec{\pi}_M \wedge \vec{B}_M) + \frac{e}{J_M} \cdot \vec{E}_M + \frac{1}{K_M} \cdot \vec{\Delta}_4$$

$$-\frac{e}{J_M^3} \cdot \vec{E}_M^\dagger \cdot \vec{\pi}_M \cdot \vec{\pi}_M - \frac{1}{K_M} \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_4 \cdot \vec{v}_M + \vec{\Delta}_5 . \tag{B.7}$$

---

[41] The partial derivative $\partial/\partial t$ in (B.6) acts on functions depending on $\vec{r}_M, t, \vec{p}_M$.

To eliminate $\vec{\pi}_M$ from the rhs of (B.7) I use (B.4) to calculate:

$$\vec{v}_M^\dagger \cdot \vec{v}_M = \frac{1}{J_M^2} \cdot \vec{\pi}_M^\dagger \cdot \vec{\pi}_M + \frac{2}{J_M} \cdot \vec{\pi}_M^\dagger \cdot \vec{\Delta}_3 = \frac{1}{J_M^2} \cdot \vec{\pi}_M^\dagger \cdot \vec{\pi}_M + 2 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3$$

$$= 1 - \frac{m^2}{J_M^2} + 2 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3 , \tag{B.8}$$

so that

$$J_M^2 = m^2 \cdot \left(1 - \vec{v}_M^\dagger \cdot \vec{v}_M + 2 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3\right)^{-1} = m^2 \cdot (1 - \vec{v}_M^\dagger \cdot \vec{v}_M)^{-1} \cdot \left(1 + \frac{2 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3}{1 - \vec{v}_M^\dagger \cdot \vec{v}_M}\right)^{-1}$$

$$= m^2 \cdot \gamma_M^2 \cdot \left(1 - 2 \cdot \gamma_M^2 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3\right) ,$$

$$J_M = m \cdot \gamma_M \cdot \left(1 - \gamma_M^2 \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_3\right) = K_M - \Delta_6 ,$$

$$\frac{1}{J_M} = (K_M - \Delta_6)^{-1} = \frac{1}{K_M} \cdot (1 - \Delta_6/K_M)^{-1} = \frac{1}{K_M} + \frac{\Delta_6}{K_M^2} ,$$

$$\frac{1}{J_M^2} = \frac{1}{K_M^2} + \frac{2 \cdot \Delta_6}{K_M^3} ,$$

$$\frac{\vec{\pi}_M}{J_M^2} = \frac{\vec{v}_M}{J_M} - \frac{\vec{\Delta}_3}{J_M} = \vec{v}_M \cdot \left[\frac{1}{K_M} + \frac{\Delta_6}{K_M^2}\right] - \frac{\vec{\Delta}_3}{K_M} . \tag{B.9}$$

Inserting (B.4),(B.9) into (B.7) yields

$$\vec{v}_M' = \left(\frac{e}{K_M} + \frac{e \cdot \Delta_6}{K_M^2}\right) \cdot (\vec{v}_M \wedge \vec{B}_M) - \frac{e}{K_M} \cdot (\vec{\Delta}_3 \wedge \vec{B}_M) + \left(\frac{e}{K_M} + \frac{e \cdot \Delta_6}{K_M^2}\right) \cdot \vec{E}_M + \frac{1}{K_M} \cdot \vec{\Delta}_4$$

$$-e \cdot \vec{E}_M^\dagger \cdot [\vec{v}_M - \vec{\Delta}_3] \cdot [\vec{v}_M \cdot \left(\frac{1}{K_M} + \frac{\Delta_6}{K_M^2}\right) - \frac{\vec{\Delta}_3}{K_M}] - \frac{1}{K_M} \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_4 \cdot \vec{v}_M + \vec{\Delta}_5$$

$$= \frac{e}{K_M} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{K_M} \cdot \vec{E}_M - \frac{e}{K_M} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{v}_M + \frac{e}{K_M^2} \cdot \Delta_6 \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$-\frac{e}{K_M} \cdot (\vec{\Delta}_3 \wedge \vec{B}_M) + \frac{e}{K_M^2} \cdot \Delta_6 \cdot \vec{E}_M + \frac{1}{K_M} \cdot \vec{\Delta}_4 + \frac{e}{K_M} \cdot \vec{E}_M^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_M$$

$$-\frac{e}{K_M^2} \cdot \Delta_6 \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{v}_M + \frac{e}{K_M} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{\Delta}_3 - \frac{1}{K_M} \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_4 \cdot \vec{v}_M + \vec{\Delta}_5 . \tag{B.10}$$

B.3

Now I begin to evaluate $m \cdot (\gamma_P \cdot \vec{v}_P)'$ and it is clear by (2.3b) that:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = m \cdot \gamma_P \cdot \vec{v}_P' + m \cdot \gamma_P^3 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P' \cdot \vec{v}_P . \tag{B.11}$$

The remaining task in this Appendix is to reexpress the rhs of (B.11) in terms of $\vec{r}_P, t, \vec{v}_P, \vec{s}$ in order to demonstrate that it is identical with the rhs of (2.10).

Introducing the abbreviations:

$$\vec{\Delta}_7 \equiv \vec{v}_P - \vec{v}_M ,$$
$$\vec{\Delta}_8 \equiv \vec{\Delta}_7' , \tag{B.12}$$

one gets by (1.3),(1.6),(2.3a),(2.5),(A.1):

$$\vec{\Omega}_P^\dagger \cdot \vec{v}_P = -\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \ ,$$

$$\vec{\Omega}_M^\dagger \cdot \vec{v}_M = -\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_M} \cdot \vec{B}_M^\dagger \cdot \vec{v}_M \ ,$$

$$\vec{\Delta}_7 = \left(\frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M)\right)' = \left(\frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{\pi}_M)\right)'$$

$$= -\frac{1}{m^2} \cdot \gamma_M' \cdot \frac{1}{(\gamma_M + 1)^2} \cdot (\vec{\sigma} \wedge \vec{\pi}_M) + \frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{\pi}_M)'$$

$$= -\frac{e}{m^3} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \frac{1}{(\gamma_M + 1)^2} \cdot (\vec{\sigma} \wedge \vec{\pi}_M) + \frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{\pi}_M)'$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \frac{\gamma_M}{(\gamma_M + 1)^2} \cdot (\vec{\sigma} \wedge \vec{v}_M) + \frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{\pi}_M)'$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \frac{\gamma_M}{(\gamma_M + 1)^2} \cdot (\vec{\sigma} \wedge \vec{v}_M) + \frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left((\vec{\Omega}_M \wedge \vec{\sigma}) \wedge \vec{\pi}_M + \vec{\sigma} \wedge \vec{\pi}_M'\right)$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \frac{1}{(\gamma_M + 1)^2} \cdot (\vec{s} \wedge \vec{v}_M) + \frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left(\vec{\Omega}_M^\dagger \cdot \vec{\pi}_M \cdot [\frac{1}{\gamma_M} \cdot \vec{s}\right.$$

$$\left. + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M] - \vec{s}^\dagger \cdot \vec{\pi}_M \cdot \vec{\Omega}_M + [\frac{1}{\gamma_M} \cdot \vec{s} + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M] \wedge \vec{\pi}_M'\right)$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \frac{1}{(\gamma_M + 1)^2} \cdot (\vec{s} \wedge \vec{v}_M)$$

$$- \frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \vec{B}_M^\dagger \cdot \vec{v}_M \cdot [\frac{1}{\gamma_M} \cdot \vec{s} + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M] - \frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{\Omega}_M$$

$$+ \frac{1}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot [\frac{1}{\gamma_M} \cdot \vec{s} + \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M] \wedge [e \cdot (\vec{v}_M \wedge \vec{B}_M) + e \cdot \vec{E}_M]$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)$$

$$- \frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{B}_M^\dagger \cdot \vec{v}_P \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] - \frac{1}{m} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{\Omega}_P$$

$$+ \frac{1}{m^2} \cdot \frac{1}{\gamma_P + 1} \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \wedge [e \cdot (\vec{v}_P \wedge \vec{B}_M) + e \cdot \vec{E}_M]$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)$$

$$- \frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{B}_M^\dagger \cdot \vec{v}_P \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]$$

$$+ \frac{e}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \left([\frac{1}{\gamma_P} + \frac{g - 2}{2}] \cdot \vec{B}_M - \frac{g - 2}{2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_M \cdot \vec{v}_P\right.$$

$$\left. - [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_M)\right) + \frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{B}_M \cdot \vec{v}_P$$

$$- \frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_M + \frac{e}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{B}_M \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P$$

$$- \frac{e}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_M + \frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot (\vec{s} \wedge \vec{E}_M)$$

$$+\frac{e}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{E}_M)$$

$$= -\frac{e}{m^2} \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{B}_M^\dagger \cdot \vec{v}_P \cdot \vec{s}$$

$$-\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{B}_M^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_M$$

$$+\frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{B}_M \cdot \vec{v}_P - \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{E}_M)$$

$$+\frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot (\vec{s} \wedge \vec{E}_M) \ . \tag{B.13}$$

Next one concludes from (B.10),(B.12):

$$\vec{v}_P' = \vec{v}_M' + \vec{\Delta}_8$$

$$= \ \frac{e}{K_M} \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{K_M} \cdot \vec{E}_M - \frac{e}{K_M} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{v}_M + \frac{e}{K_M^2} \cdot \Delta_6 \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$-\frac{e}{K_M} \cdot (\vec{\Delta}_3 \wedge \vec{B}_M) + \frac{e}{K_M^2} \cdot \Delta_6 \cdot \vec{E}_M + \frac{1}{K_M} \cdot \vec{\Delta}_4 + \frac{e}{K_M} \cdot \vec{E}_M^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_M$$

$$-\frac{e}{K_M^2} \cdot \Delta_6 \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{v}_M + \frac{e}{K_M} \cdot \vec{E}_M^\dagger \cdot \vec{v}_M \cdot \vec{\Delta}_3 - \frac{1}{K_M} \cdot \vec{v}_M^\dagger \cdot \vec{\Delta}_4 \cdot \vec{v}_M + \vec{\Delta}_5 + \vec{\Delta}_8 \ . \tag{B.14}$$

B.4

If one inserts (B.14) into (B.11) then the rhs of (B.11) depends explicitly on $\vec{v}_M$. In this subsection I show how the variable $\vec{v}_M$ on the rhs of (B.11),(B.14) can be replaced by $\vec{v}_P$. I abbreviate:

$$K_P \equiv m \cdot \gamma_P \ ,$$
$$\Delta_9 \equiv -K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \ . \tag{B.15}$$

From this follows by (B.1),(B.12):

$$\frac{1}{K_M^2} = \frac{1}{m^2} \cdot (1 - \vec{v}_M^\dagger \cdot \vec{v}_M) = \frac{1}{m^2} \cdot [1 - (\vec{v}_P - \vec{\Delta}_7)^\dagger \cdot (\vec{v}_P - \vec{\Delta}_7)]$$

$$= \ \frac{1}{m^2} \cdot [1 - \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7] = \frac{1}{m^2} \cdot [1 - \vec{v}_P^\dagger \cdot \vec{v}_P] \cdot [1 + 2 \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7]$$

$$= \ \frac{1}{K_P^2} \cdot [1 + 2 \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7] = \frac{1}{K_P^2} \cdot [1 - \frac{2 \cdot \Delta_9}{K_P}] = \frac{1}{K_P^2} - \frac{2 \cdot \Delta_9}{K_P^3} \ ,$$

$$\frac{1}{K_M} = \frac{1}{K_P} \cdot [1 - \frac{\Delta_9}{K_P}] = \frac{1}{K_P} - \frac{\Delta_9}{K_P^2} \ ,$$

$$K_M = K_P + \Delta_9 \ ,$$

$$\frac{\vec{v}_M}{K_M} = [\frac{1}{K_P} - \frac{\Delta_9}{K_P^2}] \cdot [\vec{v}_P - \vec{\Delta}_7] = \frac{1}{K_P} \cdot \vec{v}_P - \frac{\Delta_9}{K_P^2} \cdot \vec{v}_P - \frac{1}{K_P} \cdot \vec{\Delta}_7 \ . \tag{B.16}$$

Also one has by (2.5),(B.1),(B.3),(B.12-13),(B.15):

$$\vec{\Delta}_3 = \frac{e}{K_P^2} \cdot \vec{v}_P \cdot \vec{B}_M^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] + \frac{e}{(K_P + m)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_M) \cdot \vec{v}_P$$

$$-\frac{e}{m\cdot(K_P+m)}\cdot\vec{E}_M\wedge[\frac{1}{\gamma_P}\cdot\vec{s}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]$$

$$+\frac{e\cdot g}{2\cdot m\cdot K_P}\cdot\vec{E}_M\wedge[\frac{1}{\gamma_P}\cdot\vec{s}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]-\frac{e\cdot g}{2\cdot K_P^2}\cdot\vec{s}^\dagger\cdot(\vec{v}_P\wedge\vec{E}_M)\cdot\vec{v}_P$$

$$+\frac{g-2}{2}\cdot\left(\frac{e}{m}\cdot\frac{1}{K_P+m}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{B}_M-\frac{e}{m}\cdot\frac{m+2\cdot K_P}{(K_P+m)^2}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_M\cdot\vec{v}_P\right.$$

$$\left.+\frac{e}{m}\cdot\frac{1}{K_P+m}\cdot\vec{v}_P^\dagger\cdot\vec{B}_M\cdot[\frac{1}{\gamma_P}\cdot\vec{s}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]\right)$$

$$=\quad\frac{e}{K_P^2}\cdot\vec{v}_P\cdot\vec{B}_M^\dagger\cdot[\frac{1}{\gamma_P}\cdot\vec{s}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]+\frac{e}{(K_P+m)^2}\cdot\vec{s}^\dagger\cdot(\vec{v}_P\wedge\vec{E}_M)\cdot\vec{v}_P$$

$$-\frac{e}{m\cdot(K_P+m)}\cdot\vec{E}_M\wedge[\frac{1}{\gamma_P}\cdot\vec{s}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]$$

$$+\frac{e\cdot g}{2\cdot m\cdot K_P}\cdot\vec{E}_M\wedge[\frac{1}{\gamma_P}\cdot\vec{s}+\frac{\gamma_P}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]-\frac{e\cdot g}{2\cdot K_P^2}\cdot\vec{s}^\dagger\cdot(\vec{v}_P\wedge\vec{E}_M)\cdot\vec{v}_P$$

$$+\frac{g-2}{2}\cdot\left(\frac{e}{m}\cdot\frac{1}{K_P+m}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{B}_M-\frac{e}{m^2}\cdot\frac{1}{\gamma_P+1}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_M\cdot\vec{v}_P\right.$$

$$\left.+\frac{e}{m^2}\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)}\cdot\vec{v}_P^\dagger\cdot\vec{B}_M\cdot\vec{s}\right),$$

$$\vec{\Delta}_3+\vec{\Delta}_7=\frac{e}{m^2}\cdot\frac{\gamma_P^2+\gamma_P+1}{\gamma_P^3\cdot(\gamma_P+1)}\cdot\vec{B}_M^\dagger\cdot\vec{s}\cdot\vec{v}_P$$

$$+\frac{e}{m^2}\cdot[\frac{1}{\gamma_P\cdot(\gamma_P+1)}-\frac{g-2}{2}]\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_M\cdot\vec{v}_P$$

$$+\frac{e}{m^2}\cdot[\frac{1}{(\gamma_P+1)^2}-\frac{g}{2\cdot\gamma_P^2}]\cdot\vec{s}^\dagger\cdot(\vec{v}_P\wedge\vec{E}_M)\cdot\vec{v}_P$$

$$+\frac{e}{m^2}\cdot[-\frac{2}{\gamma_P\cdot(\gamma_P+1)}+\frac{g}{2\cdot\gamma_P^2}]\cdot(\vec{E}_M\wedge\vec{s})$$

$$+\frac{e}{m^2}\cdot[-\frac{\gamma_P^2+2\cdot\gamma_P}{(\gamma_P+1)^2}+\frac{g}{2}]\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot(\vec{E}_M\wedge\vec{v}_P)+\frac{e\cdot(g-2)}{2\cdot m^2}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{B}_M$$

$$-\frac{e}{m^2}\cdot\vec{E}_M^\dagger\cdot\vec{v}_P\cdot\frac{1}{(\gamma_P+1)^2}\cdot(\vec{s}\wedge\vec{v}_P)-\frac{e}{m^2}\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)}\cdot\vec{B}_M^\dagger\cdot\vec{v}_P\cdot\vec{s},$$

$$\Delta_6=m\cdot\gamma_P^3\cdot\vec{v}_P^\dagger\cdot\vec{\Delta}_3\,.\tag{B.17}$$

Combining (B.16-17) I can replace on the rhs of (B.14) the variable $\vec{v}_M$ by $\vec{v}_P$:

$$\vec{v}_P'=[\frac{e}{K_P}\cdot\vec{v}_P-\frac{e\cdot\Delta_9}{K_P^2}\cdot\vec{v}_P-\frac{e}{K_P}\cdot\vec{\Delta}_7]\wedge\vec{B}_M+[\frac{e}{K_P}-\frac{e\cdot\Delta_9}{K_P^2}]\cdot\vec{E}_M$$

$$-e\cdot\vec{E}_M^\dagger\cdot[\frac{1}{K_P}\cdot\vec{v}_P-\frac{\Delta_9}{K_P^2}\cdot\vec{v}_P-\frac{1}{K_P}\cdot\vec{\Delta}_7]\cdot[\vec{v}_P-\vec{\Delta}_7]+\frac{e}{K_P^2}\cdot\Delta_6\cdot(\vec{v}_P\wedge\vec{B}_M)$$

$$-\frac{e}{K_P}\cdot(\vec{\Delta}_3\wedge\vec{B}_M)+\frac{e}{K_P^2}\cdot\Delta_6\cdot\vec{E}_M+\frac{1}{K_P}\cdot\vec{\Delta}_4+\frac{e}{K_P}\cdot\vec{E}_M^\dagger\cdot\vec{\Delta}_3\cdot\vec{v}_P$$

$$-\frac{e}{K_P^2}\cdot\Delta_6\cdot\vec{E}_M^\dagger\cdot\vec{v}_P\cdot\vec{v}_P+\frac{e}{K_P}\cdot\vec{E}_M^\dagger\cdot\vec{v}_P\cdot\vec{\Delta}_3-\frac{1}{K_P}\cdot\vec{v}_P^\dagger\cdot\vec{\Delta}_4\cdot\vec{v}_P+\vec{\Delta}_5+\vec{\Delta}_8$$

$$=\quad\frac{e}{K_P}\cdot(\vec{v}_P\wedge\vec{B}_M)+\frac{e}{K_P}\cdot\vec{E}_M-\frac{e}{K_P}\cdot\vec{E}_M^\dagger\cdot\vec{v}_P\cdot\vec{v}_P-\frac{e\cdot\Delta_9}{K_P^2}\cdot(\vec{v}_P\wedge\vec{B}_M)$$

$$-\frac{e}{K_P} \cdot (\vec{\Delta}_7 \wedge \vec{B}_M) - \frac{e \cdot \Delta_9}{K_P^2} \cdot \vec{E}_M + \frac{e}{K_P} \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \vec{\Delta}_7$$

$$+\frac{e}{K_P^2} \cdot \Delta_9 \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \frac{e}{K_P} \cdot \vec{E}_M^\dagger \cdot \vec{\Delta}_7 \cdot \vec{v}_P + \frac{e}{K_P^2} \cdot \Delta_6 \cdot (\vec{v}_P \wedge \vec{B}_M)$$

$$-\frac{e}{K_P} \cdot (\vec{\Delta}_3 \wedge \vec{B}_M) + \frac{e}{K_P^2} \cdot \Delta_6 \cdot \vec{E}_M + \frac{1}{K_P} \cdot \vec{\Delta}_4 + \frac{e}{K_P} \cdot \vec{E}_M^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_P$$

$$-\frac{e}{K_P^2} \cdot \Delta_6 \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \frac{e}{K_P} \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \vec{\Delta}_3 - \frac{1}{K_P} \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_4 \cdot \vec{v}_P + \vec{\Delta}_5 + \vec{\Delta}_8 \; ,$$

$$\text{(B.18)}$$

so that (B.11) reads as:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = m \cdot \gamma_P \cdot \vec{v}_P' + m \cdot \gamma_P^3 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P' \cdot \vec{v}_P$$

$$= e \cdot (\vec{v}_P \wedge \vec{B}_M) + e \cdot \vec{E}_M$$

$$+\frac{e}{K_P} \cdot [\Delta_6 - \Delta_9] \cdot [\vec{v}_P \wedge \vec{B}_M + \vec{E}_M] - e \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [(\vec{\Delta}_3 + \vec{\Delta}_7) \wedge \vec{B}_M]$$

$$+e \cdot [\vec{E}_M^\dagger \cdot \vec{v}_P + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_M^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_M^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7]$$

$$+\vec{\Delta}_4 + K_P \cdot [\vec{\Delta}_5 + \vec{\Delta}_8] + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot [\vec{\Delta}_5 + \vec{\Delta}_8] \cdot \vec{v}_P \; . \qquad \text{(B.19)}$$

B.5

In this subsection I show how the variable $\vec{r}_M$ on the rhs of (B.19) can be replaced by $\vec{r}_P$. The dependence on $\vec{r}_M$ comes in only via the field vectors $\vec{E}_M, \vec{B}_M$ and their first derivatives. First of all by using (2.1),(2.5) I abbreviate

$$\vec{\Delta}_{10} \equiv \vec{r}_M - \vec{r}_P = -\frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M) = -\frac{1}{m} \cdot \frac{1}{\gamma_M + 1} \cdot (\vec{s} \wedge \vec{v}_M)$$

$$= -\frac{1}{m} \cdot \frac{1}{\gamma_P + 1} \cdot (\vec{s} \wedge \vec{v}_P) \; ,$$

$$\vec{\Delta}_{11} \equiv (\vec{\Delta}_{10}^\dagger \cdot \vec{\nabla}_M)\vec{E}_M \; ,$$

$$\vec{\Delta}_{12} \equiv (\vec{\Delta}_{10}^\dagger \cdot \vec{\nabla}_M)\vec{B}_M \; , \qquad \text{(B.20)}$$

from which follows:

$$\vec{E}_M = \vec{E}_M(\vec{r}_M, t) = \vec{E}_M(\vec{r}_P + \vec{\Delta}_{10}, t) = \vec{E}_M(\vec{r}_P, t) + \vec{\Delta}_{11} = \vec{E}_P + \vec{\Delta}_{11} \; ,$$

$$\vec{B}_M = \vec{B}_M(\vec{r}_M, t) = \vec{B}_M(\vec{r}_P + \vec{\Delta}_{10}, t) = \vec{B}_M(\vec{r}_P, t) + \vec{\Delta}_{12} = \vec{B}_P + \vec{\Delta}_{12} \; . \qquad \text{(B.21)}$$

Inserting this into (B.19) yields:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = \underbrace{e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P}_{\text{Lorentz force}} + \underbrace{e \cdot (\vec{v}_P \wedge \vec{\Delta}_{12}) + e \cdot \vec{\Delta}_{11}}_{\text{first order SG terms}}$$

$$\underbrace{+\vec{\Delta}_4 + K_P \cdot [\vec{\Delta}_5 + \vec{\Delta}_8] + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot [\vec{\Delta}_5 + \vec{\Delta}_8] \cdot \vec{v}_P}_{\text{first order and second order SG terms}}$$

$$\underbrace{+\frac{e}{K_P} \cdot [\Delta_6 - \Delta_9] \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] - e \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [(\vec{\Delta}_3 + \vec{\Delta}_7) \wedge \vec{B}_P]}_{\text{second order SG terms}}$$

$$\underbrace{+e \cdot [\vec{E}_P^\dagger \cdot \vec{v}_P + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7]}_{\text{second order SG terms}} \; . \qquad \text{(B.22)}$$

Introducing the abbreviation

$$
\vec{\Delta}_{13} \equiv e \cdot (\vec{v}_P \wedge \vec{\Delta}_{12}) + e \cdot \vec{\Delta}_{11} + \vec{\Delta}_4 + K_P \cdot [\vec{\Delta}_5 + \vec{\Delta}_8] + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot [\vec{\Delta}_5 + \vec{\Delta}_8] \cdot \vec{v}_P
$$
$$
+ \frac{e}{K_P} \cdot [\Delta_6 - \Delta_9] \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] - e \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [(\vec{\Delta}_3 + \vec{\Delta}_7) \wedge \vec{B}_P]
$$
$$
+ e \cdot [\vec{E}_P^\dagger \cdot \vec{v}_P + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] , \tag{B.23}
$$

one then gets

$$
m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \vec{\Delta}_{13} . \tag{B.24}
$$

I now have to show that (B.24) is identical with (2.10). Therefore the remaining task of this Appendix is to simplify $\vec{\Delta}_{13}$. The first two terms on the rhs of (B.24) constitute the Lorentz force whereas the remaining part constitutes the SG force. To disentangle $\vec{\Delta}_{13}$ it is important to notice that the SG force occurs in two different forms. The 'first order part' contains the field vectors $\vec{E}_P, \vec{B}_P$ only linearly; more specifically it is linear in the first derivatives of the field vectors. The 'second order part' contains the field vectors $\vec{E}_P, \vec{B}_P$ quadratically. Note that in the second order part no derivatives of the field vectors occur. In equation (B.22) I have indicated which of the two forms of the SG force occurs in a term.

Accordingly one can split the SG force $\vec{\Delta}_{13}$ into a first order part plus a second order part. First I abbreviate by using (2.8): [42]

$$
\vec{\Delta}_{14} \equiv -\vec{\nabla}_P(\vec{\sigma}^\dagger \cdot \vec{\Omega}_P) = -\vec{\nabla}_P(\vec{\Omega}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P])
$$
$$
= -\frac{1}{\gamma_P} \cdot \vec{\nabla}_P(\vec{\Omega}_P^\dagger \cdot \vec{s}) + \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{\nabla}_P(\vec{B}_P^\dagger \cdot \vec{v}_P)
$$
$$
= \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P\left(\vec{s}^\dagger \cdot \left[[\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P)]\right)\right)
$$
$$
= \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \left([\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P - [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P)\right))
$$
$$
+ \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \left((\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P + \vec{v}_P \wedge \frac{\partial \vec{E}_P}{\partial t}\right) ,
$$
$$
\vec{\Delta}_{15} \equiv \frac{e}{K_P^2} \cdot \vec{v}_P \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]^\dagger \cdot \vec{B}_P' + \frac{e}{(K_P + m)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P
$$
$$
- \frac{e}{m \cdot (K_P + m)} \cdot \vec{E}_P' \wedge [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]
$$
$$
+ \frac{e \cdot g}{2 \cdot m \cdot K_P} \cdot \vec{E}_P' \wedge [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] - \frac{e \cdot g}{2 \cdot K_P^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P
$$
$$
+ \frac{g-2}{2} \cdot \left(\frac{e}{m} \cdot \frac{1}{K_P + m} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' - \frac{e}{m^2} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P\right)
$$
$$
+ \frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}\right) ,
$$
$$
\vec{\Delta}_{16} \equiv -\frac{e}{m^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}
$$

[42]The partial derivative $\partial/\partial t$ in (B.25), (B.30) and (B.32) acts on a function depending on $\vec{r}_P, t$.

$$-\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P'$$

$$+\frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P - \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{E}_P')$$

$$+\frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot (\vec{s} \wedge \vec{E}_P') \, ,$$

$$\vec{\Delta}_{17} \equiv \vec{\Delta}_4 - \vec{\Delta}_{14} \, ,$$

$$\vec{\Delta}_{18} \equiv \vec{\Delta}_5 - \vec{\Delta}_{15} \, ,$$

$$\vec{\Delta}_{19} \equiv \vec{\Delta}_8 - \vec{\Delta}_{16} \, , \tag{B.25}$$

from which follows

$$\vec{\Delta}_4 = \vec{\Delta}_{14} + \vec{\Delta}_{17} \, ,$$
$$\vec{\Delta}_5 = \vec{\Delta}_{15} + \vec{\Delta}_{18} \, ,$$
$$\vec{\Delta}_8 = \vec{\Delta}_{16} + \vec{\Delta}_{19} \, . \tag{B.26}$$

Note that $\vec{\Delta}_{14}, \vec{\Delta}_{15}, \vec{\Delta}_{16}$ are linear in the electromagnetic field vectors, whereas $\vec{\Delta}_{17}, \vec{\Delta}_{18}, \vec{\Delta}_{19}$ are quadratic. With (B.25-26) I can now abbreviate

$$\vec{\Delta}_{20} \equiv e \cdot (\vec{v}_P \wedge \vec{\Delta}_{12}) + e \cdot \vec{\Delta}_{11}$$
$$+\vec{\Delta}_{14} + K_P \cdot [\vec{\Delta}_{15} + \vec{\Delta}_{16}] + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot [\vec{\Delta}_{15} + \vec{\Delta}_{16}] \cdot \vec{v}_P \, , \tag{B.27a}$$

which denotes the first order part of the SG force and

$$\vec{\Delta}_{21} \equiv \vec{\Delta}_{17} + K_P \cdot [\vec{\Delta}_{18} + \vec{\Delta}_{19}] + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot [\vec{\Delta}_{18} + \vec{\Delta}_{19}] \cdot \vec{v}_P$$
$$+\frac{e}{K_P} \cdot [\Delta_6 - \Delta_9] \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] - e \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [(\vec{\Delta}_3 + \vec{\Delta}_7) \wedge \vec{B}_P]$$
$$+e \cdot [\vec{E}_P^\dagger \cdot \vec{v}_P + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{E}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \, , \tag{B.27b}$$

which denotes the second order part of the SG force. Then

$$\vec{\Delta}_{13} = \vec{\Delta}_{20} + \vec{\Delta}_{21} \, . \tag{B.28}$$

Thus (B.24) reads as:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \underbrace{\vec{\Delta}_{20}}_{\text{first order SG terms}} + \underbrace{\vec{\Delta}_{21}}_{\text{second order SG terms}} \, . \tag{B.29}$$

B.6

In this subsection I simplify the first order part $\vec{\Delta}_{20}$ of the SG force. First of all I calculate by using (2.8),(B.20):

$$\vec{\nabla}_P \wedge (\vec{v}_P \wedge \vec{B}_P) = -(\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P \, ,$$

$$\vec{\nabla}_P (\vec{v}_P^\dagger \cdot \vec{B}_P) = (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P + \vec{v}_P \wedge (\vec{\nabla}_P \wedge \vec{B}_P) = (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P + \vec{v}_P \wedge \frac{\partial \vec{E}_P}{\partial t} \, ,$$

$$e \cdot (\vec{v}_P \wedge \vec{\Delta}_{12}) = e \cdot \vec{v}_P \wedge [(\vec{\Delta}_{10}^\dagger \cdot \vec{\nabla}_P)\vec{B}_P] = e \cdot (\vec{\Delta}_{10}^\dagger \cdot \vec{\nabla}_P)(\vec{v}_P \wedge \vec{B}_P)$$

49

$$= e \cdot \vec{\nabla}_P[\vec{\Delta}_{10}^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P)] - e \cdot \vec{\Delta}_{10} \wedge [\vec{\nabla}_P \wedge (\vec{v}_P \wedge \vec{B}_P)]$$

$$= -\frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{\nabla}_P((\vec{s} \wedge \vec{v}_P)^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P)) + e \cdot \vec{\Delta}_{10}^\dagger \wedge [(\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P]$$

$$= -\frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P) + \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{\nabla}_P(\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P)$$
$$+ e \cdot (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)(\vec{\Delta}_{10} \wedge \vec{B}_P)$$

$$= \frac{e}{m} \cdot \frac{1}{\gamma_P + 1}\left(-\vec{\nabla}_P(\vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P) + \vec{\nabla}_P(\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P) + (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)(\vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s})\right.$$
$$\left. - (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)(\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P)\right)$$

$$= \frac{e}{m} \cdot \frac{1}{\gamma_P + 1}\left(-\vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P - \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \wedge \frac{\partial \vec{E}_P}{\partial t} + \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \vec{B}_P)\right.$$
$$\left. + (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)(\vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}) - (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)(\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P)\right) ,$$

$$e \cdot \vec{\Delta}_{11} = -\frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot [(\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{\nabla}_P]\vec{E}_P = \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \left(-\vec{\nabla}_P(\vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P))\right.$$
$$\left. + (\vec{s} \wedge \vec{v}_P) \wedge (\vec{\nabla}_P \wedge \vec{E}_P)\right)$$

$$= \frac{e}{m} \cdot \frac{1}{\gamma_P + 1}\left(-\vec{\nabla}_P[\vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)] - (\vec{s} \wedge \vec{v}_P) \wedge \frac{\partial \vec{B}_P}{\partial t}\right)$$

$$= \frac{e}{m} \cdot \frac{1}{\gamma_P + 1}\left(-\vec{\nabla}_P[\vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)] + \vec{v}_P^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{s} - \vec{s}^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{v}_P\right) . \tag{B.30}$$

Here I used the fact that the spatial derivatives of $\vec{E}_P$ and $\vec{B}_P$ only appears in the SG terms, i.e. in leading order spin. Therefore one can always approximate:

$$\frac{\partial E_{P,k}}{\partial r_{P,j}} = \frac{\partial E_{M,k}}{\partial r_{M,j}} , \qquad \frac{\partial B_{P,k}}{\partial r_{P,j}} = \frac{\partial B_{M,k}}{\partial r_{M,j}} . \qquad (j, k = 1, 2, 3)$$

Secondly I conclude from (B.25)

$$\vec{\Delta}_{15} + \vec{\Delta}_{16} = \frac{e}{m^2} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P^3 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$
$$+ \frac{e}{m^2} \cdot [\frac{1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{g-2}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$
$$+ \frac{e}{m^2} \cdot [\frac{1}{(\gamma_P + 1)^2} - \frac{g}{2 \cdot \gamma_P^2}] \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P$$
$$+ \frac{e}{m^2} \cdot [-\frac{2}{\gamma_P \cdot (\gamma_P + 1)} + \frac{g}{2 \cdot \gamma_P^2}] \cdot (\vec{E}_P' \wedge \vec{s})$$
$$+ \frac{e}{m^2} \cdot [-\frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} + \frac{g}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P' \wedge \vec{v}_P) + \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' \cdot \vec{v}_P$$
$$- \frac{e}{m^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} ,$$

$$K_P \cdot [\vec{\Delta}_{15} + \vec{\Delta}_{16}] + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot [\vec{\Delta}_{15} + \vec{\Delta}_{16}] \cdot \vec{v}_P = \frac{e}{m} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$-\frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma + 1)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P$$

$$+\frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+\frac{e}{m} \cdot [-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P' \wedge \vec{v}_P) + \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P'$$

$$-\frac{e}{m} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} \,. \tag{B.31}$$

Combining (B.25),(B.27),(B.30-31) one gets

$$\vec{\Delta}_{20} = \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \Big(-\vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P - \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \frac{\partial \vec{E}_P}{\partial t}) + \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \vec{B}_P)$$

$$+ (\vec{v}_P^\dagger \cdot \vec{\nabla}_P)(\vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}) - \vec{v}_P^\dagger \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P)\Big)$$

$$+ \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \Big(-\vec{\nabla}_P[\vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)] + \vec{v}_P^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{s} - \vec{s}^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{v}_P \Big)$$

$$+ \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \Big([\frac{1}{\gamma_P} + \frac{g - 2}{2}] \cdot \vec{B}_P - [\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P)\Big))$$

$$+ \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \Big((\vec{v}_P^\dagger \cdot \vec{\nabla}_P)\vec{B}_P + \vec{v}_P \wedge \frac{\partial \vec{E}_P}{\partial t}\Big)$$

$$+ \frac{e}{m} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$- \frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma + 1)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P$$

$$+ \frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+ \frac{e}{m} \cdot [-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P' \wedge \vec{v}_P) + \frac{e \cdot (g - 2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P'$$

$$- \frac{e}{m} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}$$

$$= \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P(\vec{s}^\dagger \cdot \vec{B}_P) + \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \Big(\vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} - \vec{v}_P^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{s} - \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$+ \vec{s}^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{v}_P \Big) + \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \Big(-\vec{\nabla}_P[\vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)] + \vec{v}_P^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{s} - \vec{s}^\dagger \cdot \frac{\partial \vec{B}_P}{\partial t} \cdot \vec{v}_P \Big)$$

$$+ \frac{e}{m} \cdot [\frac{1}{\gamma_P + 1} - \frac{g}{2 \cdot \gamma_P}] \cdot \vec{\nabla}_P[\vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P)]$$

$$+ \frac{e}{m} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$- \frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma + 1)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P$$

$$+ \frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+\frac{e}{m} \cdot [-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P' \wedge \vec{v}_P) + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P'$$

$$-\frac{e}{m} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}$$

$$= \quad \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P (\vec{s}^\dagger \cdot \vec{B}_P) + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' + \frac{e}{m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$-\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P [\vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P)] - \frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma + 1)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P$$

$$+\frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot (\vec{E}_P' \wedge \vec{s})$$

$$-\frac{e}{m} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+\frac{e}{m} \cdot [-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P' \wedge \vec{v}_P) . \tag{B.32}$$

This can be further simplified by calculating

$$\vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{s} \wedge \vec{v}_P) - \vec{v}_P^\dagger \cdot \vec{v}_P \cdot (\vec{s} \wedge \vec{E}_P') = \vec{s} \wedge (\vec{v}_P \wedge (\vec{v}_P \wedge \vec{E}_P'))$$

$$= \vec{v}_P \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') - \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{E}_P') , \tag{B.33}$$

from which follows

$$\vec{\Delta}_{20} = \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P (\vec{s}^\dagger \cdot \vec{B}_P) + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' + \frac{e}{m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$-\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{\nabla}_P [\vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P)]$$

$$-\frac{e \cdot g}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P') \cdot \vec{v}_P$$

$$+\frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P} - \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P] \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+\frac{e}{m} \cdot [-\frac{\gamma_P}{(\gamma_P + 1)^2} - \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{s} \wedge \vec{v}_P)$$

$$= \quad \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right) + \frac{e \cdot \gamma_P}{2 \cdot m} \cdot \left(g \cdot [\vec{s}^\dagger \cdot \vec{B}_P' - (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P'] \cdot \vec{v}_P \right.$$

$$\left. +(g-2) \cdot [\vec{E}_P' + \vec{v}_P \wedge \vec{B}_P' - \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P] \wedge \vec{s} \right) . \tag{B.34}$$

Inserting (B.34) into (B.29) one has thus obtained:

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+\frac{e \cdot \gamma_P}{2 \cdot m} \cdot \left(g \cdot [\vec{s}^\dagger \cdot \vec{B}_P' - (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P'] \cdot \vec{v}_P + (g-2) \cdot [\vec{E}_P' + \vec{v}_P \wedge \vec{B}_P' - \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P] \wedge \vec{s} \right)$$

$$+\vec{\Delta}_{21} . \tag{B.35}$$

With (B.35) I have simplified the Lorentz and the first order SG terms and have derived the first order terms of (2.10). [43] In the above derivation it was essential that the electromagnetic

---

[43]One sees by (B.35) that by neglecting second order SG terms the charge e appears only up to first order. Thus one could have derived the first order SG terms in an alternative way by making first order perturbation

field obeys (2.8), i.e. is a solution of the vacuum Maxwell equations. The second order SG terms are simplified below.

B.7

In the remaining subsections of this Appendix I complete the derivation of (2.10) by disentangling $\vec{\Delta}_{21}$, i.e. I have to deal with the second order SG terms. [44]

First of all I simplify the rhs of (B.23) by collecting its terms in a convenient way and to do this I calculate by using (2.12),(B.5),(B.12):

$$
\left(K_P \cdot [\vec{\Delta}_3 + \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_P] \right)' = \frac{1}{m} \cdot \left(K_P \cdot [m \cdot \vec{\Delta}_3 + m \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_P] \right)'
$$
$$
= \frac{1}{m} \cdot K_P' \cdot [m \cdot \vec{\Delta}_3 + m \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_P] + \frac{1}{m} \cdot K_P \cdot [m \cdot \vec{\Delta}_5
$$
$$
+ m \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_5 \cdot \vec{v}_P + m \cdot \gamma_P \cdot \vec{\Delta}_3^\dagger \cdot (\gamma_P \cdot \vec{v}_P)' \cdot \vec{v}_P + m \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_3 \cdot (\gamma_P \cdot \vec{v}_P)']
$$
$$
= e \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{\Delta}_3 + e \cdot \gamma_P^2 \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_3 \cdot \vec{v}_P + K_P \cdot \vec{\Delta}_5 + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_5 \cdot \vec{v}_P
$$
$$
+ e \cdot \gamma_P^2 \cdot \vec{\Delta}_3^\dagger \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] \cdot \vec{v}_P + e \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_3 \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] \,,
$$
$$
\left(K_P \cdot [\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \cdot \vec{v}_P] \right)' = \frac{1}{m} \cdot \left(K_P \cdot [m \cdot \vec{\Delta}_7 + m \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \cdot \vec{v}_P] \right)'
$$
$$
= \frac{1}{m} \cdot K_P' \cdot [m \cdot \vec{\Delta}_7 + m \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \cdot \vec{v}_P] + \frac{1}{m} \cdot K_P \cdot [m \cdot \vec{\Delta}_8
$$
$$
+ m \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_8 \cdot \vec{v}_P + m \cdot \gamma_P \cdot \vec{\Delta}_7^\dagger \cdot (\gamma_P \cdot \vec{v}_P)' \cdot \vec{v}_P + m \cdot \gamma_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \cdot (\gamma_P \cdot \vec{v}_P)']
$$
$$
= e \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{\Delta}_7 + e \cdot \gamma_P^2 \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \cdot \vec{v}_P + K_P \cdot \vec{\Delta}_8 + K_P \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_8 \cdot \vec{v}_P
$$
$$
+ e \cdot \gamma_P^2 \cdot \vec{\Delta}_7^\dagger \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] \cdot \vec{v}_P + e \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] \,. \tag{B.36}
$$

Thus (B.23) can be rewritten as:

$$
\vec{\Delta}_{13} = \underbrace{e \cdot (\vec{v}_P \wedge \vec{\Delta}_{12}) + e \cdot \vec{\Delta}_{11}}_{\text{first order SG terms}} + \underbrace{\vec{\Delta}_4 + \left(K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \right)'}_{\text{first order and second order SG terms}}
$$
$$
- \underbrace{e \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \wedge \vec{B}_P}_{\text{second order SG terms}} \,. \tag{B.37}
$$

To obtain $\vec{\Delta}_{21}$, i.e. to identify the second order SG terms on the rhs of (B.37), one first observes by (B.25) that $\vec{\Delta}_4$ contains the second order term $\vec{\Delta}_{17}$. Also one has by (B.5),(B.12):

$$
\left(K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \right)' = K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_5 + \vec{\Delta}_8]
$$
$$
+ \left(K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \right)' \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \,.
$$

On the rhs of this equation only the first part contains first order SG terms and by (B.25) one observes that the first order terms are given by $K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_{15} + \vec{\Delta}_{16}]$. Introducing

---

theory w.r.t. the charge. This approach is chosen in [DS70], so that from this point of view the first 6 subsections of Appendix B are just a check of [DS70].

[44]To the knowledge of the author this is the first treatment which takes the second order SG force into account.

the abbreviations:

$$\vec{\Delta}_{22} \equiv \left(K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \right)' - K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_{15} + \vec{\Delta}_{16}] \; ,$$

$$\vec{\Delta}_{23} \equiv -e \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] \wedge \vec{B}_P \; , \tag{B.38}$$

one thus can simplify the second order SG terms as follows:

$$\vec{\Delta}_{21} = \vec{\Delta}_{17} + \vec{\Delta}_{22} + \vec{\Delta}_{23} \; . \tag{B.39}$$

B.8

In this subsection I simplify $\vec{\Delta}_{22}$ and I first of all calculate

$$\begin{aligned}
\vec{\Delta}_3 + \vec{\Delta}_7 &= \frac{e}{m^2} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P^3 \cdot (\gamma_P + 1)} \cdot \vec{B}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P \\
&\quad + \frac{e}{m^2} \cdot [\frac{1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{g-2}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \\
&\quad + \frac{e}{m^2} \cdot [\frac{1}{(\gamma_P + 1)^2} - \frac{g}{2 \cdot \gamma_P^2}] \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P) \cdot \vec{v}_P \\
&\quad + \frac{e}{m^2} \cdot [-\frac{2}{\gamma_P \cdot (\gamma_P + 1)} + \frac{g}{2 \cdot \gamma_P^2}] \cdot (\vec{E}_P \wedge \vec{s}) \\
&\quad + \frac{e}{m^2} \cdot [-\frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} + \frac{g}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P) + \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P \\
&\quad - \frac{e}{m^2} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{B}_M^\dagger \cdot \vec{v}_P \cdot \vec{s} \; ,
\end{aligned}$$

$$\begin{aligned}
K_P \cdot [1 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger] \cdot [\vec{\Delta}_3 + \vec{\Delta}_7] &= \frac{e}{m} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \\
&\quad - \frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma + 1)^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P) \cdot \vec{v}_P \\
&\quad + \frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot (\vec{E}_P \wedge \vec{s}) \\
&\quad + \frac{e}{m} \cdot [-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P) + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P \\
&\quad - \frac{e}{m} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} \; , \tag{B.40}
\end{aligned}$$

from which follows by (B.38):

$$\begin{aligned}
\vec{\Delta}_{22} &= \frac{e}{m} \cdot \left(\frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{v}_P \cdot \vec{s}^\dagger \right)' \cdot \vec{B}_P - \frac{e}{m} \cdot \left(\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} \cdot \vec{v}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \right)' \cdot \vec{E}_P \\
&\quad - \frac{e}{m} \cdot \left([-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot \vec{s} \right)' \wedge \vec{E}_P \\
&\quad - \frac{e}{m} \cdot \left([-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \right)' \wedge \vec{E}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \left(\gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \right)' \cdot \vec{B}_P \\
&\quad - \frac{e}{m} \cdot \left(\frac{\gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{v}_P^\dagger \right)' \cdot \vec{E}_P - \frac{e}{m} \cdot \left(\frac{1}{\gamma_P + 1} \cdot \vec{s} \cdot \vec{v}_P^\dagger \right)' \cdot \vec{B}_P \; . \tag{B.41}
\end{aligned}$$

To simplify this, I calculate by (2.3a),(2.5),(2.9),(B.12-13),(B.18):

$$\vec{B}_P^\dagger \cdot \vec{s}\,' = \vec{B}_P^\dagger \cdot \left[\frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \left(\vec{s} \wedge \vec{B}_P + (\vec{v}_P \wedge \vec{s}) \wedge \vec{E}_P \right) \right.$$

$$\left. + \frac{e \cdot (g-2)}{2 \cdot m \cdot \gamma_P} \cdot \left(\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{E}_P + \vec{v}_P \wedge \vec{B}_P] - \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s} - \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P \right) \right]$$

$$= \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \left(\vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{s} - \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \right) + \frac{e \cdot (g-2)}{2 \cdot m \cdot \gamma_P} \cdot \left(\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \right.$$

$$\left. - \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P - \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \right)$$

$$= -\frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \left(\vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \right.$$

$$\left. - \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \right) + \frac{e}{m \cdot \gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \ ,$$

$$\left(\vec{s} \wedge \vec{v}_P \right)' = \left(\gamma_P \cdot (\vec{\sigma} \wedge \vec{v}_P) \right)' = \left(\gamma_M \cdot (\vec{\sigma} \wedge \vec{v}_M) \right)' = m \cdot (\gamma_M + 1) \cdot \vec{\Delta}_7$$

$$- \gamma_M \cdot (\gamma_M + 1) \cdot (\vec{\sigma} \wedge \vec{v}_M) \cdot \left(\frac{1}{\gamma_M + 1} \right)'$$

$$= m \cdot (\gamma_M + 1) \cdot \vec{\Delta}_7 + \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{E}_P \cdot (\vec{\sigma} \wedge \vec{v}_M)$$

$$= m \cdot (\gamma_P + 1) \cdot \vec{\Delta}_7 + \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)$$

$$= -\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}$$

$$- \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$+ \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+ \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{E}_P) \ ,$$

$$\vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)' = -\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{E}_P$$

$$- \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{E}_P$$

$$+ \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \ ,$$

$$\vec{E}_P \wedge \vec{s}\,' = \vec{E}_P \wedge \left[\frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \left(\vec{s} \wedge \vec{B}_P + (\vec{v}_P \wedge \vec{s}) \wedge \vec{E}_P \right) \right.$$

$$\left. + \frac{e \cdot (g-2)}{2 \cdot m \cdot \gamma_P} \cdot \left(\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{E}_P + \vec{v}_P \wedge \vec{B}_P] - \vec{v}^\dagger \cdot \vec{E}_P \cdot \vec{s} - \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P \right) \right]$$

$$= \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \left(\vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P + \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P \right)$$

$$+ \frac{e \cdot (g-2)}{2 \cdot m \cdot \gamma_P} \cdot \left(\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P \right.$$

<div align="center">55</div>

$$-\vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) - \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P) \Big) ,$$

$$(\vec{s}^\dagger \cdot \vec{v}_P)' = (\vec{\sigma}^\dagger \cdot \vec{v}_P)' = \vec{v}_P^\dagger \cdot (\vec{\Omega}_M \wedge \vec{\sigma}) + \vec{\sigma}^\dagger \cdot \vec{v}_P'$$

$$= \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{\Omega}_P + \frac{e}{K_P} \cdot \vec{\sigma}^\dagger \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P - \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]$$

$$= -\frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \left([\frac{1}{\gamma_P} + \frac{g-2}{2}] \cdot \vec{B}_P - \frac{g-2}{2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \right.$$

$$\left. -[\frac{g}{2} - \frac{\gamma_P}{\gamma_P + 1}] \cdot (\vec{v}_P \wedge \vec{E}_P) \right)$$

$$+\frac{e}{m} \cdot \frac{1}{\gamma_P^2} \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e}{K_P} \cdot \vec{E}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]$$

$$-\frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P$$

$$= -\frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P$$

$$+\frac{e}{m} \cdot [\frac{1}{\gamma_P} - \frac{g}{2} \cdot \frac{\gamma_P^2 - 1}{\gamma_P^3}] \cdot \vec{E}_P^\dagger \cdot \vec{s} ,$$

$$s_j \cdot \vec{B}_P^\dagger \cdot \vec{v}_P' = s_j \cdot \left(\frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{B}_P^\dagger \cdot \vec{E}_P - \frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \right) , \quad (j = 1, 2, 3) \quad (B.42)$$

from which follows by (B.41),(2.9):

$$\vec{\Delta}_{22} = \frac{e^2}{m^3} \cdot \frac{\gamma_P \cdot (\gamma_P + 1) \cdot (2 \cdot \gamma_P + 1) - (\gamma_P^2 + \gamma_P + 1) \cdot (2 \cdot \gamma_P + 1)}{\gamma_P^2 \cdot (\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{\pi}_M$$

$$+\frac{e}{m^2} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{\pi}_M' + \frac{e}{m} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{s}\,'$$

$$-\frac{e^2}{m^3} \cdot \frac{(2 \cdot \gamma_P + 2) \cdot (\gamma_P + 1)^2 - 2 \cdot (\gamma_P + 1) \cdot (\gamma_P^2 + 2 \cdot \gamma_P)}{(\gamma_P + 1)^4} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{\pi}_M$$

$$-\frac{e}{m^2} \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{\pi}_M' - \frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)'$$

$$-\frac{e^2}{m^2} \cdot [\frac{2}{(\gamma_P + 1)^2} - \frac{g}{2 \cdot \gamma_P^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P) + \frac{e}{m} \cdot [-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P}] \cdot \vec{E}_P \wedge \vec{s}\,'$$

$$+\frac{e^2}{m^3} \cdot \frac{(\gamma_P + 1)^2 \cdot (2 \cdot \gamma_P + 2) - 2 \cdot (\gamma_P + 1) \cdot (\gamma_P^2 + 2 \cdot \gamma_P)}{(\gamma_P + 1)^4} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{\pi}_M \wedge \vec{E}_P)$$

$$-\frac{e}{m^2} \cdot [-\frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} + \frac{g}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{\pi}_M' \wedge \vec{E}_P$$

$$-\frac{e}{m} \cdot [-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P] \cdot (\vec{s}^\dagger \cdot \vec{v}_P)' \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+\frac{e^2 \cdot (g-2)}{2 \cdot m^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot (\vec{s}^\dagger \cdot \vec{v}_P)' \cdot \vec{B}_P$$

$$+\frac{2 \cdot e^2}{m^3} \cdot \frac{1}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{\pi}_M^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P) - \frac{e}{m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{\pi}_M' \cdot (\vec{s} \wedge \vec{v}_P)$$

$$-\frac{e}{m} \cdot \frac{\gamma_P}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)' + \frac{e^2}{m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$-\frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P' - \frac{e}{m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}\,'$$

$$= -\frac{e^2}{m^2} \cdot \frac{2 \cdot \gamma_P + 1}{\gamma_P \cdot (\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + \frac{e^2}{m^2} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P]$$

$$+\frac{e}{m} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P + 1} \cdot \vec{v}_P \cdot \left[-\frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \right.$$

$$+\frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \left(\vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P - \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \right)$$

$$+\frac{e}{m \cdot \gamma_P} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \Bigg] - \frac{e^2}{m^2} \cdot \frac{2 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{v}_P$$

$$-\frac{e^2}{m^2} \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P]$$

$$-\frac{e}{m} \cdot \frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} \cdot \vec{v}_P \cdot \left(-\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{E}_P \right.$$

$$-\frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{E}_P$$

$$+\frac{e}{m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \right) - \frac{e^2}{m^2} \cdot \left[\frac{2}{(\gamma_P + 1)^2} - \frac{g}{2 \cdot \gamma_P^2} \right] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P)$$

$$+\frac{e}{m} \cdot \left[-\frac{2}{\gamma_P + 1} + \frac{g}{2 \cdot \gamma_P} \right] \cdot \left[\frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \left(\vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P \right. \right.$$

$$\vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P \Big) + \frac{e \cdot (g-2)}{2 \cdot m \cdot \gamma_P} \cdot \left(\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \right.$$

$$\left. -\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P - \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) - \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P) \right) \Big]$$

$$+\frac{e^2}{m^2} \cdot \frac{2 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$-\frac{e^2}{m^2} \cdot \left[-\frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} + \frac{g}{2} \right] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{B}_P - \vec{B}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P]$$

$$-\frac{e}{m} \cdot \left[-\frac{\gamma_P^3 + 2 \cdot \gamma_P^2}{(\gamma_P + 1)^2} + \frac{g}{2} \cdot \gamma_P \right] \cdot (\vec{v}_P \wedge \vec{E}_P) \cdot \left(-\frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \right.$$

$$+\frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P + \frac{e}{m} \cdot \left[\frac{1}{\gamma_P} - \frac{g}{2} \cdot \frac{\gamma_P^2 - 1}{\gamma_P^3} \right] \cdot \vec{E}_P^\dagger \cdot \vec{s} \right)$$

$$+\frac{e^2 \cdot (g-2)}{2 \cdot m^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$+\frac{e \cdot (g-2)}{2 \cdot m} \cdot \gamma_P \cdot \vec{B}_P \cdot \left(-\frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \right.$$

$$+\frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{1}{\gamma_P} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P + \frac{e}{m} \cdot \left[\frac{1}{\gamma_P} - \frac{g}{2} \cdot \frac{\gamma_P^2 - 1}{\gamma_P^3} \right] \cdot \vec{E}_P^\dagger \cdot \vec{s} \right)$$

$$+\frac{2 \cdot e^2}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{E}_P^\dagger \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P]$$

$$-\frac{e}{m}\cdot\frac{\gamma_P}{(\gamma_P+1)^2}\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\left(-\frac{e\cdot g}{2\cdot m}\cdot\frac{1}{\gamma_P}\cdot\vec{B}_P^\dagger\cdot\vec{v}_P\cdot\vec{s}\right.$$

$$-\frac{e\cdot(g-2)}{2\cdot m}\cdot\gamma_P\cdot\vec{B}_P^\dagger\cdot\vec{v}_P\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P+\frac{e\cdot(g-2)}{2\cdot m}\cdot\gamma_P\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{B}_P$$

$$+\frac{e}{m}\cdot\frac{1}{\gamma_P}\cdot\vec{s}^\dagger\cdot\vec{B}_P\cdot\vec{v}_P-\frac{e\cdot(g-2)}{2\cdot m}\cdot\gamma_P\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot(\vec{v}_P\wedge\vec{E}_P)+\frac{e}{m}\cdot\frac{1}{\gamma_P}\cdot(\vec{s}\wedge\vec{E}_P)\Big)$$

$$+\frac{e^2}{m^2}\cdot\frac{1}{(\gamma_P+1)^2}\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot\vec{s}$$

$$-\frac{e}{m}\cdot\frac{1}{\gamma_P+1}\cdot\vec{s}\cdot\left(\frac{e}{m}\cdot\frac{1}{\gamma_P}\cdot\vec{B}_P^\dagger\cdot\vec{E}_P-\frac{e}{m}\cdot\frac{1}{\gamma_P}\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\right)$$

$$-\frac{e}{m}\cdot\frac{1}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot\left[\frac{e\cdot g}{2\cdot m\cdot\gamma_P}\cdot\left(\vec{s}\wedge\vec{B}_P+(\vec{v}_P\wedge\vec{s})\wedge\vec{E}_P\right)\right.$$

$$\left.+\frac{e\cdot(g-2)}{2\cdot m\cdot\gamma_P}\cdot\left(\gamma_P^2\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot[\vec{E}_P+\vec{v}_P\wedge\vec{B}_P]-\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\vec{s}-\gamma_P^2\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\vec{v}_P\right)\right]$$

$$=\ -\frac{e^2}{m^2}\cdot\frac{1}{(\gamma_P+1)^2}\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot\vec{s}^\dagger\cdot\vec{B}_P\cdot\vec{v}_P+\frac{e^2}{m^2}\cdot\frac{\gamma_P^2+\gamma_P+1}{\gamma_P\cdot(\gamma_P+1)}\cdot\vec{s}^\dagger\cdot\vec{B}_P\cdot[\vec{v}_P\wedge\vec{B}_P+\vec{E}_P]$$

$$-\frac{e^2\cdot g}{2\cdot m^2}\cdot\frac{1}{(\gamma_P+1)^2}\cdot\vec{E}_P^\dagger\cdot\vec{s}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot\vec{v}_P$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}-\frac{g}{2}\cdot\frac{\gamma_P}{(\gamma_P+1)^2}-\frac{\gamma_P}{\gamma_P+1}]\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{E}_P^\dagger\cdot\vec{B}_P\cdot\vec{v}_P$$

$$-\frac{e^2}{m^2}\cdot\frac{2\cdot\gamma_P}{(\gamma_P+1)^3}\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot(\vec{s}\wedge\vec{v}_P)^\dagger\cdot\vec{E}_P\cdot\vec{v}_P$$

$$-\frac{e^2}{m^2}\cdot\frac{\gamma_P^2+2\cdot\gamma_P}{(\gamma_P+1)^2}\cdot(\vec{s}\wedge\vec{v}_P)^\dagger\cdot\vec{E}_P\cdot(\vec{v}_P\wedge\vec{B}_P)$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}\cdot\frac{1}{\gamma_P^2}-g\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)}-\frac{\gamma_P^2+2\cdot\gamma_P}{(\gamma_P+1)^2}]\cdot\vec{E}_P^\dagger\cdot(\vec{s}\wedge\vec{v}_P)\cdot\vec{E}_P$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}\cdot\frac{1}{\gamma_P^2}-g\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)}-\frac{\gamma_P-2}{\gamma_P\cdot(\gamma_P+1)^2}]\cdot\vec{v}_P^\dagger\cdot\vec{E}_P\cdot(\vec{s}\wedge\vec{E}_P)$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}\cdot\frac{1}{\gamma_P^2}-g\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)}-\frac{1}{\gamma_P\cdot(\gamma_P+1)}]\cdot\vec{E}_P^\dagger\cdot\vec{B}_P\cdot\vec{s}$$

$$+\frac{e^2}{m^2}\cdot[-\frac{g^2}{4}+\frac{g}{2}\cdot\frac{2\cdot\gamma_P^3+2\cdot\gamma_P^2+\gamma_P-1}{\gamma_P^2\cdot(\gamma_P+1)}-1]\cdot\vec{s}^\dagger\cdot\vec{E}_P\cdot\vec{B}_P$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}\cdot\frac{1}{\gamma_P^2}-g\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)}+\frac{1}{(\gamma_P+1)^2}]\cdot\vec{E}_P^\dagger\cdot\vec{E}_P\cdot(\vec{v}_P\wedge\vec{s})$$

$$-\frac{e^2\cdot g}{2\cdot m^2}\cdot\frac{1}{(1+\gamma_P)^2}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{E}_P^\dagger\cdot\vec{v}_P\cdot\vec{B}_P$$

$$-\frac{2\cdot e^2}{m^2}\cdot\frac{\gamma_P}{(\gamma_P+1)^3}\cdot\vec{s}^\dagger\cdot\vec{v}_P\cdot\vec{E}_P^\dagger\cdot\vec{v}_P\cdot(\vec{E}_P\wedge\vec{v}_P)$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}-\frac{g}{2}\cdot\frac{2\cdot\gamma_P^2+4\cdot\gamma_P+1}{(\gamma_P+1)^2}+\frac{\gamma_P^2+2\cdot\gamma_P}{(\gamma_P+1)^2}]\cdot(\vec{s}\wedge\vec{v}_P)^\dagger\cdot\vec{B}_P\cdot(\vec{v}_P\wedge\vec{E}_P)$$

$$+\frac{e^2}{m^2}\cdot[\frac{g^2}{4}\cdot\frac{\gamma_P^2-1}{\gamma_P^2}-g\cdot\frac{\gamma_P^2+\gamma_P-1}{\gamma_P\cdot(\gamma_P+1)}+\frac{\gamma_P^2+2\cdot\gamma_P}{(\gamma_P+1)^2}]\cdot\vec{E}_P^\dagger\cdot\vec{s}\cdot(\vec{v}_P\wedge\vec{E}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{(g-2)^2}{4} \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{B}_P$$

$$+\frac{2 \cdot e^2}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g}{2} \cdot \frac{1}{(\gamma_P + 1)^2} + \frac{1}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$-\frac{e^2 \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{B}_P)$$

$$-\frac{e^2 \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot [\vec{E}_P + \vec{v}_P \wedge \vec{B}_P] \ . \tag{B.43}$$

B.9

In this subsection I simplify $\vec{\Delta}_{17}$ and $\vec{\Delta}_{23}$ and use:

$$\{\pi_{M,j}, \pi_{M,m}\}_M = e \cdot \sum_{n=1}^3 \varepsilon_{jmn} \cdot B_{M,n} \ ,$$

$$\sigma_k \cdot \{\pi_{M,j}, \pi_{M,m}\}_M = e \cdot \sigma_k \cdot \sum_{n=1}^3 \varepsilon_{jmn} \cdot B_{P,n} \ , \qquad (j, k, m = 1, 2, 3)$$

which follows from (1.1),(1.2),(B.21). Thus I get by (1.3),(B.5),(B.25):

$$\vec{\Delta}_{17} = \vec{\Delta}_4 - \vec{\Delta}_{14} = \{\vec{\pi}_M, \vec{\sigma}^\dagger \cdot \vec{W}_M\}_M - \vec{\Delta}_{14}$$

$$= -\frac{e}{m} \cdot \left(\{\vec{\pi}_M, \frac{m}{J_M}\}_M \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P - \frac{g-2}{2} \cdot \{\vec{\pi}_M, \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{\sigma} \cdot \vec{\pi}_M^\dagger\}_M \cdot \vec{B}_P \right.$$

$$\left. -\{\vec{\pi}_M, [\frac{g}{2 \cdot J_M} - \frac{1}{J_M + m}] \cdot (\vec{\sigma} \wedge \vec{\pi}_M)^\dagger\}_M \cdot \vec{E}_P \right)$$

$$= \frac{e^2}{J_M^2} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \frac{e^2}{m} \cdot \frac{g-2}{2} \cdot \frac{2 \cdot J_M + m}{J_M^2 \cdot (J_M + m)^2} \cdot \vec{\pi}_M^\dagger \cdot \vec{\sigma} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+\frac{e^2}{m} \cdot \frac{g-2}{2} \cdot \frac{1}{J_M \cdot (J_M + m)} \cdot \vec{\pi}_M^\dagger \cdot \vec{B}_P \cdot (\vec{\sigma} \wedge \vec{B}_P)$$

$$+\frac{e^2}{m} \cdot [-\frac{g}{2 \cdot J_M^2} + \frac{1}{(J_M + m)^2}] \cdot (\vec{\sigma} \wedge \vec{\pi}_M)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+\frac{e^2}{m} \cdot [\frac{g}{2 \cdot J_M} - \frac{1}{J_M + m}] \cdot [(\vec{E}_P \wedge \vec{\sigma}) \wedge \vec{B}_P]$$

$$= \frac{e^2}{m^2} \cdot \frac{1}{\gamma_P^2} \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot \vec{B}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]$$

$$-\frac{e^2}{m^2} \cdot \frac{g-2}{2} \cdot \frac{2 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{g-2}{2} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{B}_P \wedge [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P]$$

$$+\frac{e^2}{m^2} \cdot [-\frac{g}{2 \cdot \gamma_P} + \frac{\gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{\sigma} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g}{2 \cdot \gamma_P} - \frac{1}{\gamma_P + 1}] \cdot \left(\vec{E}_P^\dagger \cdot \vec{B}_P \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \right.$$

$$\left. -\vec{E}_P \cdot \vec{B}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \right)$$

$$= \quad \frac{e^2}{m^2} \cdot \frac{1}{\gamma_P^3} \cdot \vec{B}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e^2}{m^2} \cdot [\frac{1}{\gamma_P} - \frac{g}{2} \cdot \frac{1}{\gamma_P + 1}] \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{g-2}{2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s})$$

$$+\frac{e^2}{m^2} \cdot [-\frac{g}{2 \cdot \gamma_P^2} + \frac{1}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g}{2 \cdot \gamma_P} - \frac{1}{\gamma_P + 1}] \cdot \left(\vec{E}_P^\dagger \cdot \vec{B}_P \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \right.$$

$$\left. -\vec{E}_P \cdot \vec{B}_P^\dagger \cdot [\frac{1}{\gamma_P} \cdot \vec{s} + \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P] \right) . \tag{B.44}$$

Next I calculate by using (B.17),(B.38):

$$\vec{\Delta}_{23} = \left(-\frac{e^2}{m^2} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P^3 \cdot (\gamma_P + 1)} \cdot \vec{B}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P \right.$$

$$-\frac{e^2}{m^2} \cdot [\frac{1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{g-2}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$-\frac{e^2}{m^2} \cdot [\frac{1}{(\gamma_P + 1)^2} - \frac{g}{2 \cdot \gamma_P^2}] \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P) \cdot \vec{v}_P$$

$$-\frac{e^2}{m^2} \cdot [-\frac{2}{\gamma_P \cdot (\gamma_P + 1)} + \frac{g}{2 \cdot \gamma_P^2}] \cdot (\vec{E}_P \wedge \vec{s})$$

$$-\frac{e^2}{m^2} \cdot [-\frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} + \frac{g}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P) - \frac{e^2 \cdot (g-2)}{2 \cdot m^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$+\frac{e^2}{m^2} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \frac{1}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+\left. \frac{e^2}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s} \right) \wedge \vec{B}_P$$

$$= \quad -\frac{e^2}{m^2} \cdot \frac{\gamma_P^2 + \gamma_P + 1}{\gamma_P^3 \cdot (\gamma_P + 1)} \cdot \vec{B}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$-\frac{e^2}{m^2} \cdot [\frac{1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{g-2}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$-\frac{e^2}{m^2} \cdot [\frac{1}{(\gamma_P + 1)^2} - \frac{g}{2 \cdot \gamma_P^2}] \cdot \vec{s}^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P) \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$-\frac{e^2}{m^2} \cdot [-\frac{2}{\gamma_P \cdot (\gamma_P + 1)} + \frac{g}{2 \cdot \gamma_P^2}] \cdot [\vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - \vec{B}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P]$$

$$-\frac{e^2}{m^2} \cdot [-\frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} + \frac{g}{2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P]$$

$$+\frac{e^2}{m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot [\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}]$$

60

$$+\frac{e^2}{m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot (\vec{s} \wedge \vec{B}_P) \,. \tag{B.45}$$

B.10

Combining (B.39),(B.43-45) the second order SG terms read as follows:

$$\vec{\Delta}_{21} = \vec{\Delta}_{17} + \vec{\Delta}_{22} + \vec{\Delta}_{23}$$

$$= \frac{e^2}{m^2} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] - \frac{e^2 \cdot g}{2 \cdot m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} - \frac{g}{2} \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$-\frac{e^2}{m^2} \cdot \frac{2 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{v}_P$$

$$-\frac{e^2}{m^2} \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} \cdot \frac{1}{\gamma_P^2} - g \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{E}_P$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} \cdot \frac{1}{\gamma_P^2} - g \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{\gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} \cdot \frac{1}{\gamma_P^2} - g \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+\frac{e^2}{m^2} \cdot [-\frac{g^2}{4} + \frac{g}{2} \cdot \frac{2 \cdot \gamma_P^3 + 2 \cdot \gamma_P^2 + \gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} - 1] \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} \cdot \frac{1}{\gamma_P^2} - g \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} + \frac{1}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$-\frac{e^2 \cdot g}{2 \cdot m^2} \cdot \frac{1}{(1 + \gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$-\frac{2 \cdot e^2}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} - \frac{g}{2} \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+\frac{e^2}{m^2} \cdot [\frac{g^2}{4} \cdot \frac{\gamma_P^2 - 1}{\gamma_P^2} - g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} + \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{(g - 2)^2}{4} \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{B}_P$$

$$+\frac{2 \cdot e^2}{m^2} \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)$$

$$-\frac{e^2}{m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+\frac{e^2 \cdot g}{2 \cdot m^2} \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} \,. \tag{B.46}$$

With (B.46) I have an explicit form of the second order SG terms and my remaining task of this Appendix is to reduce them to those in (2.10). I abbreviate the second order SG terms of (2.10) by

$$\vec{\Delta}_{24} \equiv \frac{e^2}{4 \cdot m^2} \cdot [-(g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) + (g-2)^2 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$-(g-2)^2 \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P + (-g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 4) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$-(g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) + (g-2)^2 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+(g-2)^2 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - (g^2 - 4 \cdot g) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+(g-2) \cdot g \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+(g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$-(g^2 - 4 \cdot g) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot g \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] , \qquad (B.47)$$

Therefore the remaining task of this Appendix is to show that the rhs of (B.46) equals $\vec{\Delta}_{24}$.

Introducing the abbreviation

$$\vec{\Delta}_{25} \equiv \vec{\Delta}_{21} - \vec{\Delta}_{24} = \frac{e^2}{4 \cdot m^2} \cdot \left((g-2)^2 \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) \right.$$

$$+(g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$-2 \cdot g \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + [g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$-\frac{8 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{v}_P$$

$$+[2 \cdot g - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+[g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{E}_P$$

$$+[g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P)$$

$$+[g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P$$

$$+[g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$-2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P - 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P)$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+[g^2 \cdot \frac{\gamma_P^2 - 1}{\gamma_P^2} - 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$-(g-2)^2 \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{B}_P + 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+[g^2 - 4 \cdot g + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+(g - 2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - (g - 2)^2 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$-(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P \Big) , \tag{B.48}$$

I have to show that $\vec{\Delta}_{25}$ vanishes. One can separate $\vec{\Delta}_{25}$ into a magnetic part $\vec{\Delta}_{26}$ plus an electric part $\vec{\Delta}_{27}$ plus a mixed part $\vec{\Delta}_{28}$. Therefore I abbreviate:

$$\vec{\Delta}_{25} \equiv \vec{\Delta}_{26} + \vec{\Delta}_{27} + \vec{\Delta}_{28} , \tag{B.49}$$

where

$$\vec{\Delta}_{26} \equiv \frac{e^2}{4 \cdot m^2} \cdot \Big((g - 2)^2 \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - (g - 2)^2 \cdot \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{B}_P$$

$$+ (g - 2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - (g - 2)^2 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) \Big)$$

$$= \frac{e^2 \cdot (g - 2)^2}{4 \cdot m^2} \cdot \Big(\vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{B}_P + \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s})$$

$$- \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) \Big) ,$$

$$\vec{\Delta}_{27} \equiv \frac{e^2}{4 \cdot m^2} \cdot \Big(-\frac{8 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{v}_P$$

$$+ [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{E}_P$$

$$+ [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P)$$

$$+ [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$- 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P)$$

$$+ [g^2 \cdot \frac{\gamma_P^2 - 1}{\gamma_P^2} - 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+ 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P) \Big) ,$$

$$\vec{\Delta}_{28} \equiv \frac{e^2}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- 2 \cdot g \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + [g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$+ [2 \cdot g - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+ [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P$$

$$- 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+[g^2 - 4 \cdot g + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)$$

$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$-(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P \Big) . \tag{B.50}$$

Hence the remaining task of this Appendix is to show that $\vec{\Delta}_{26}, \vec{\Delta}_{27}, \vec{\Delta}_{28}$ vanish.

B.11

In this subsection I simplify $\vec{\Delta}_{26}$. I use the same method as in section A.2. If $\vec{v}_P, \vec{B}_P$ are linearly independent, I have the following 3 linearly independent vectors:

$$\vec{v}_P, \vec{B}_P, \vec{v}_P \wedge \vec{B}_P .$$

One concludes from (B.50):

$$\vec{v}_P^\dagger \cdot \vec{\Delta}_{26} = 0 ,$$
$$\vec{B}_P^\dagger \cdot \vec{\Delta}_{26} = 0 ,$$
$$(\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{\Delta}_{26} = \frac{e^2 \cdot (g - 2)^2}{4 \cdot m^2} \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \Big(\vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \vec{B}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{B}_P$$

$$+\vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) \Big) = \frac{e^2 \cdot (g - 2)^2}{4 \cdot m^2} \cdot [\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$

$$-\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P + \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P - \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P$$

$$-\vec{B}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{s} + \vec{B}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P] = 0 , \tag{B.51}$$

so that $\vec{\Delta}_{26}$ vanishes, if $\vec{v}_P, \vec{B}_P$ are linearly independent.

If $\vec{v}_P, \vec{B}_P$ are linearly dependent, then $\vec{v}_P \wedge \vec{B}_P$ vanishes so that one gets from (B.50):

$$\vec{\Delta}_{26} = \frac{e^2 \cdot (g - 2)^2}{4 \cdot m^2} \cdot \Big(\vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) \Big) = 0 . \tag{B.52}$$

One observes by (B.52) that $\vec{\Delta}_{26}$ vanishes if $\vec{v}_P = 0$ or $\vec{B}_P = 0$. It remains to consider the subcase with: $\vec{B}_P = \lambda \cdot \vec{v}_P$, where λ is a constant which balances the dimensions. Then the rhs of (B.52) vanishes. Therefore $\vec{\Delta}_{26}$ vanishes in any case.

B.12

In this subsection I simplify $\vec{\Delta}_{27}$. If $\vec{v}_P, \vec{E}_P$ are linearly independent, one has the following 3 linearly independent vectors:

$$\vec{v}_P, \vec{E}_P, \vec{v}_P \wedge \vec{E}_P .$$

One concludes from (B.50):

$$
\begin{aligned}
\vec{v}_P^\dagger \cdot \vec{\Delta}_{27} &= \frac{e^2}{4 \cdot m^2} \cdot \Big(-\frac{8 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \\
&\quad + [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \\
&\quad + [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} \\
&\quad\quad - \frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P)^\dagger \cdot \vec{v}_P \Big) \\
&= \frac{e^2}{4 \cdot m^2} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)^3} \cdot [-8 \cdot \gamma_P^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \\
&\quad - (4 \cdot \gamma_P^2 + 8 \cdot \gamma_P) \cdot (\gamma_P^2 + \gamma_P) + (4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8) \cdot (\gamma_P + 1)] = 0 \ , \\
\vec{E}_P^\dagger \cdot \vec{\Delta}_{27} &= \frac{e^2}{4 \cdot m^2} \cdot \Big(-\frac{8 \cdot \gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \\
&\quad + [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \\
&\quad + [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})^\dagger \cdot \vec{E}_P \\
&\quad + 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \Big) = 0 \ , \\
(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{27} &= \frac{e^2}{4 \cdot m^2} \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \Big([g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} \\
&\quad - \frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P) \\
&\quad + [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) \\
&\quad - 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P \wedge \vec{v}_P) \\
&\quad + [g^2 \cdot \frac{\gamma_P^2 - 1}{\gamma_P^2} - 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P) \\
&\quad + 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P) \Big) \\
&= \frac{e^2}{4 \cdot m^2} \cdot \Big([-\frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2} \\
&\quad + 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} + [\frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2} \\
&\quad - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2} - 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} \Big) = 0 \ ,
\end{aligned}
$$

$$(B.53)$$

so that $\vec{\Delta}_{27}$ vanishes, if $\vec{v}_P, \vec{E}_P$ are linearly independent.

If $\vec{v}_P, \vec{E}_P$ are linearly dependent, then $\vec{v}_P \wedge \vec{E}_P$ vanishes so that one gets from (B.50):

$$\vec{\Delta}_{27} = \frac{e^2}{4 \cdot m^2} \cdot \Big([g^2 \cdot \frac{1-\gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)}$$
$$-\frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{E}_P) + [g^2 \cdot \frac{1-\gamma_P^2}{\gamma_P^2} + 4 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 1}{\gamma_P \cdot (\gamma_P + 1)}$$
$$-\frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) + 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{s} \wedge \vec{v}_P) \Big) .$$

$$\text{(B.54)}$$

One observes by (B.54) that $\vec{\Delta}_{27}$ vanishes if $\vec{v}_P = 0$ or $\vec{E}_P = 0$. It remains to consider the subcase with: $\vec{E}_P = \lambda \cdot \vec{v}_P$, where λ is a constant which balances the dimensions. Then (B.54) reads as:

$$\vec{\Delta}_{27} = \frac{e^2 \cdot \lambda^2}{4 \cdot m^2} \cdot (\vec{s} \wedge \vec{v}_P) \cdot \Big([-\frac{4 \cdot \gamma_P^3 + 8 \cdot \gamma_P^2 + 8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$
$$+ 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \Big)$$
$$= \frac{e^2 \cdot \lambda^2}{4 \cdot m^2} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot (\vec{s} \wedge \vec{v}_P) \cdot \Big(-\frac{8 \cdot \gamma_P - 8}{\gamma_P \cdot (\gamma_P + 1)^2} + 8 \cdot \frac{\gamma_P}{(\gamma_P + 1)^3} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \Big) = 0 .$$

Therefore $\vec{\Delta}_{27}$ vanishes in any case.

B.13

In this subsection I simplify $\vec{\Delta}_{28}$. If $\vec{v}_P, \vec{E}_P$ are linearly independent, one has the following 3 linearly independent vectors:

$$\vec{v}_P, \vec{E}_P, \vec{v}_P \wedge \vec{E}_P .$$

One concludes from (B.50):

$$\vec{v}_P^\dagger \cdot \vec{\Delta}_{28} = \frac{e^2}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P$$
$$-2 \cdot g \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$
$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$
$$+[g^2 \cdot \frac{1-\gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P$$
$$+2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P^\dagger \cdot \vec{v}_P - 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P^\dagger \cdot \vec{v}_P$$
$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P$$
$$-(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \Big) = 0 ,$$

$$\vec{E}_P^\dagger \cdot \vec{\Delta}_{28} = \frac{e^2}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$-2 \cdot g \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P$$

$$+[2 \cdot g - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P$$

$$+[g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{s}$$

$$+2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P$$

$$-2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P$$

$$+[g^2 - 4 \cdot g + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{s}$$

$$-(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \Big)$$

$$= \frac{e^2}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P + [g^2 - 2 \cdot g] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g] \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} + [g^2 - 2 \cdot g] \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P$$

$$-(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \Big) . \tag{B.55}$$

To show that $\vec{E}_P^\dagger \cdot \vec{\Delta}_{28}$ vanishes, I introduce the abbreviation [45]

$$\vec{\Delta}_{29} \equiv \vec{\nabla}_B[\vec{E}_P^\dagger \cdot \vec{\Delta}_{28}] = \frac{e^2}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}$$

$$+[g^2 - 2 \cdot g] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P + [g^2 - 2 \cdot g] \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{v}_P) + [g^2 - 2 \cdot g] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P$$

$$-(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s} - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P \Big) , \tag{B.56}$$

so that one gets

$$\vec{v}_P^\dagger \cdot \vec{\Delta}_{29} = \frac{e^2}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{s}$$

---

[45]The nabla operator $\vec{\nabla}_B$ always acts on functions depending on $t, \vec{B}_P, \vec{E}_P, \vec{v}_P, \vec{s}$ and denotes the gradient w.r.t. to $\vec{B}_P$.

$$+[g^2 - 2 \cdot g] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P + [g^2 - 2 \cdot g] \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{v}_P$$

$$+[g^2 - 2 \cdot g] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{s}$$

$$\left. -(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \right) = 0 \ ,$$

$$\vec{E}_P^\dagger \cdot \vec{\Delta}_{29} = \frac{e^2}{4 \cdot m^2} \cdot \left((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} \right.$$

$$+[g^2 - 2 \cdot g] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P + [g^2 - 2 \cdot g] \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s}$$

$$\left. -(g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \right) = 0 \ ,$$

$$(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{29} = \frac{e^2}{4 \cdot m^2} \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \left((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{s} \right.$$

$$+[g^2 - 2 \cdot g] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{v}_P) - (g - 2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s} \Big) = 0 \ . \quad \text{(B.57)}$$

Hence $\vec{\Delta}_{29}$ vanishes, if $\vec{v}_P, \vec{E}_P$ are linearly independent. Combining this with (B.56) and using the fact that $\vec{\Delta}_{28}$ is linear in $\vec{B}_P$, one concludes:

$$\vec{E}_P^\dagger \cdot \vec{\Delta}_{28} = 0 \ , \tag{B.58}$$

if $\vec{v}_P, \vec{E}_P$ are linearly independent.

Next I calculate

$$(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{28} = \frac{e^2}{4 \cdot m^2} \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \left([2 \cdot g \right.$$

$$-\frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+[g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{B}_P$$

$$-2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+[g^2 - 4 \cdot g + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{B}_P) \cdot (\vec{s} \wedge \vec{v}_P)$$

$$\left. +[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} \right)$$

$$= \frac{e^2}{4 \cdot m^2} \cdot \left([g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)} \right.$$

$$-\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s}$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s}$$

$$+[g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} - 4 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$

$$+ \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{B}_P$$

$$+[-g^2 + 4 \cdot g - 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P^\dagger \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot [\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$- \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P] \Big) . \tag{B.59}$$

To show that $(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{28}$ vanishes, I introduce the abbreviation

$$\vec{\Delta}_{30} \equiv \vec{\nabla}_B[(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{28}] = \frac{e^2}{4 \cdot m^2} \cdot \Big([g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}$$

$$- \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s} \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s} \cdot \vec{v}_P$$

$$+[g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} - 4 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$

$$+ \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+[-g^2 + 4 \cdot g - 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot [\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$- \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P] \cdot (\vec{s} \wedge \vec{v}_P) \Big) , \tag{B.60}$$

so that one gets

$$\vec{v}_P^\dagger \cdot \vec{\Delta}_{30} = \frac{e^2}{4 \cdot m^2} \cdot \Big([g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}$$

$$- \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \Big) = 0 ,$$

$$\vec{E}_P^\dagger \cdot \vec{\Delta}_{30} = \frac{e^2}{4 \cdot m^2} \cdot \Big([g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}$$

$$- \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{E}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot [\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$- \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P] \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P \Big) = 0 \ ,$$

$$(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{30} = \frac{e^2}{4 \cdot m^2} \cdot (\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \Big([g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)}$$

$$- 4 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+ [-g^2 + 4 \cdot g - 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} - \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{E}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot (\vec{v}_P \wedge \vec{E}_P)$$

$$+ [g^2 - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2} + \frac{4 \cdot \gamma_P^2 + 8 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot [\vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$- \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P] \cdot (\vec{s} \wedge \vec{v}_P) \Big)$$

$$= \frac{e^2}{4 \cdot m^2} \cdot \Big([2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} - 4 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$

$$+ 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P^\dagger \cdot \vec{E}_P$$

$$+ [-2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} + 4 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P$$

$$- 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{s}$$

$$+ [4 \cdot g - 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} - 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{s}$$

$$+ [-4 \cdot g + 2 \cdot g \cdot \frac{1}{(1 + \gamma_P)^2} + 2 \cdot g \cdot \frac{2 \cdot \gamma_P^2 + 4 \cdot \gamma_P + 1}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{s} \Big)$$

$$= 0 \ . \tag{B.61}$$

Hence $\vec{\Delta}_{30}$ vanishes, if $\vec{v}_P, \vec{E}_P$ are linearly independent. Combining this with (B.60) and using the fact that $\vec{\Delta}_{28}$ is linear in $\vec{B}_P$, one concludes:

$$(\vec{v}_P \wedge \vec{E}_P)^\dagger \cdot \vec{\Delta}_{28} = 0 \ , \tag{B.62}$$

if $\vec{v}_P, \vec{E}_P$ are linearly independent. Collecting (B.55),(B.58),(B.62) one concludes that $\vec{\Delta}_{28}$ vanishes, if $\vec{v}_P, \vec{E}_P$ are linearly independent.

To discuss the case where $\vec{v}_P, \vec{E}_P$ are linearly dependent, one first observes by (B.50) that $\vec{\Delta}_{28}$ vanishes if $\vec{v}_P = 0$ or $\vec{E}_P = 0$. It remains to consider the subcase with: $\vec{E}_P = \lambda \cdot \vec{v}_P$, where λ is a constant which balances the dimensions. Then one concludes from (B.50):

$$\vec{\Delta}_{28} = \frac{e^2 \cdot \lambda}{4 \cdot m^2} \cdot \Big((g^2 - 2 \cdot g) \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$- 2 \cdot g \cdot \frac{1}{(\gamma_P + 1)^2} \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + [g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$+ [g^2 \cdot \frac{1 - \gamma_P^2}{\gamma_P^2} + 2 \cdot g \cdot \frac{\gamma_P^2 + \gamma_P - 2}{\gamma_P \cdot (\gamma_P + 1)}] \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} + 2 \cdot g \cdot \frac{\gamma_P - 1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$-2 \cdot g \cdot \frac{1}{(1+\gamma_P)^2} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{B}_P$$

$$+[g^2 - 2 \cdot g \cdot \frac{\gamma_P^2 + 2 \cdot \gamma_P}{(\gamma_P + 1)^2}] \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$\left. -(g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P - (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \right) = 0 \,. \qquad \text{(B.63)}$$

Hence $\vec{\Delta}_{28}$ vanishes, if $\vec{v}_P, \vec{E}_P$ are linearly dependent. From this it follows that $\vec{\Delta}_{28}$ vanishes in any case.

B.14

In subsections B.11-13 I have shown that $\vec{\Delta}_{26}, \vec{\Delta}_{27}, \vec{\Delta}_{28}$ vanish so that by (B.49) it follows that:

$$\vec{\Delta}_{25} = 0 \,. \qquad \text{(B.64)}$$

Thus one has by (B.48):

$$\vec{\Delta}_{21} = \vec{\Delta}_{24} \,. \qquad \text{(B.65)}$$

Inserting (B.47),(B.65) into (B.35) results in

$$m \cdot (\gamma_P \cdot \vec{v}_P)' = e \cdot (\vec{v}_P \wedge \vec{B}_P) + e \cdot \vec{E}_P + \frac{e \cdot g}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+ \frac{e \cdot \gamma_P}{2 \cdot m} \cdot [2 \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P - g \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}_P' \cdot \vec{v}_P + (g-2) \cdot (\vec{E}_P' \wedge \vec{s})$$

$$+ (g-2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P' - (g-2) \cdot \vec{v}_P^\dagger \cdot \vec{E}_P' \cdot (\vec{v}_P \wedge \vec{s})]$$

$$+ \frac{e^2}{4 \cdot m^2} \cdot [-(g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) + (g-2)^2 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$- (g-2)^2 \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P + (-g^2 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 2 \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{v}_P + 4) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- (g-2)^2 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) + (g-2)^2 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+ (g-2)^2 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - (g^2 - 4 \cdot g) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)$$

$$+ (g-2) \cdot g \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+ (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + (g-2) \cdot g \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$- (g^2 - 4 \cdot g) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot g \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] \,. \qquad \text{(B.66)}$$

This completes the proof of (2.10).

Appendix C

In this Appendix I derive the equation of motion (8.15) for $m \cdot \gamma_M \cdot \vec{v}_M$ by using those equations in sections 1,2,5 and Appendix B which are valid for arbitrary values of $c_1, ..., c_5$. I only consider the case of static (i.e. time independent) magnetic fields and vanishing electric fields. I do this for the general case, i.e. for arbitrary values of $c_1, ..., c_5$. First of all one concludes from (B.12),(B.15-16):

$$m \cdot \gamma_M \cdot \vec{v}_M = m \cdot \gamma_P \cdot \vec{v}_P - K_P \cdot (\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7) \,. \qquad \text{(C.1)}$$

By (B.13) one has

$$\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 = -\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}$$

$$+\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P + \frac{e}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$-\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P ,$$

from which follows by (2.9),(B.42):

$$(\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7)' = -\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot (\vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s})'$$

$$+\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P)' + \frac{e}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P)'$$

$$-\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P)'$$

$$= -\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot (\vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} + \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}')$$

$$+\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \left((\vec{s}^\dagger \cdot \vec{v}_P)' \cdot \vec{B}_P + \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' \right)$$

$$+\frac{e}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P + \vec{B}_P^\dagger \cdot \vec{s}' \cdot \vec{v}_P + \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P')$$

$$-\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \left(\vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \vec{B}_P^\dagger \cdot \vec{v}_P \cdot (\vec{s}^\dagger \cdot \vec{v}_P)' \cdot \vec{v}_P + \vec{B}_P^\dagger \cdot \vec{v}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P' \right)$$

$$= -\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} + \frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s}^\dagger \cdot \vec{v}_P)' \cdot \vec{B}_P$$

$$+\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' + \frac{e}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P + \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P')$$

$$-\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \left(\vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P + \vec{B}_P^\dagger \cdot \vec{v}_P \cdot (\vec{s}^\dagger \cdot \vec{v}_P)' \cdot \vec{v}_P \right)$$

$$-\frac{e^2 \cdot g^2}{4 \cdot m^3} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \frac{e^2 \cdot g^2}{4 \cdot m^3} \cdot \frac{1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{B}_P)$$

$$= -\frac{e \cdot g}{2 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} - \frac{e^2 \cdot (g-2)^2}{4 \cdot m^3} \cdot \frac{1}{\gamma_P + 1} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot \vec{B}_P$$

$$+\frac{e \cdot (g-2)}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' + \frac{e}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P$$

$$+\frac{e^2}{m^3} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \frac{e \cdot g}{2 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P$$

$$+\frac{e^2 \cdot g \cdot (g-2)}{4 \cdot m^3} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot \vec{v}_P$$

$$-\frac{e^2 \cdot g^2}{4 \cdot m^3} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \frac{e^2 \cdot g^2}{4 \cdot m^3} \cdot \frac{1}{\gamma_P^2 \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{B}_P) .$$

One thus has:

$$\left(K_P \cdot (\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7) \right)' = K_P \cdot \left(\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7 \right)'$$

$$
\begin{aligned}
= \quad & -\frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} - \frac{e^2 \cdot (g-2)^2}{4 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot \vec{B}_P \\
& + \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' + \frac{e}{m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P \\
& + \frac{e^2}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \frac{e \cdot g}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \\
& + \frac{e^2 \cdot g \cdot (g-2)}{4 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \\
& - \frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) - \frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{B}_P) \, .
\end{aligned}
\tag{C.2}
$$

Combining this with (5.12b) yields by (C.1):

$$
\begin{aligned}
m \cdot (\gamma_M \cdot \vec{v}_M)' = \; & m \cdot (\gamma_P \cdot \vec{v}_P)' - \left(K_P \cdot (\vec{\Delta}_7 + \gamma_P^2 \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{\Delta}_7) \right)' \\
= \quad & e \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P (\vec{s}^\dagger \cdot \vec{B}_P) \\
& + \frac{e \cdot \gamma_P}{2 \cdot m} \cdot [(c_1 + 2) \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P + (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P'] \\
& + \frac{e^2}{4 \cdot m^2} \cdot [c_5 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - c_5 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) \\
& + (2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P)] + \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} \\
& + \frac{e^2 \cdot (g-2)^2}{4 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot \vec{B}_P \\
& - \frac{e \cdot (g-2)}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{B}_P' - \frac{e}{m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P \\
& - \frac{e^2}{m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot g}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \\
& - \frac{e^2 \cdot g \cdot (g-2)}{4 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{B}_P \cdot \vec{v}_P \\
& + \frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \frac{\gamma_P}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{s} \wedge \vec{B}_P) \\
= \quad & e \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m \cdot \gamma_P} \cdot \vec{\nabla}_P (\vec{s}^\dagger \cdot \vec{B}_P) \\
& + \frac{e}{2 \cdot m} \cdot \vec{s}^\dagger \cdot \vec{B}_P' \cdot \vec{v}_P \cdot \left(\frac{2 \cdot \gamma_P}{\gamma_P + 1} + c_1 \cdot \gamma_P \right) \\
& + \frac{e}{2 \cdot m} \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P' \cdot \gamma_P \cdot \left(c_2 - c_1 - \frac{2}{\gamma_P + 1} - g \cdot \frac{\gamma_P}{\gamma_P + 1} \right) + \frac{e \cdot g}{2 \cdot m} \cdot \frac{1}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s} \\
& + \frac{e \cdot g}{2 \cdot m} \cdot \frac{\gamma_P^2}{\gamma_P + 1} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P' \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P \\
& + \frac{e^2}{4 \cdot m^2} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) \cdot \left(c_5 - g^2 \cdot \frac{1}{\gamma_P \cdot (\gamma_P + 1)} \right)
\end{aligned}
$$

$$-\frac{e^2}{4\cdot m^2}\cdot c_5\cdot \vec{B}_P^\dagger\cdot \vec{B}_P\cdot(\vec{v}_P\wedge\vec{s})$$

$$+\frac{e^2}{4\cdot m^2}\cdot \vec{s}^\dagger\cdot \vec{B}_P\cdot(\vec{v}_P\wedge\vec{B}_P)\cdot(2\cdot c_4-c_3-\frac{4\cdot\gamma_P}{\gamma_P+1})$$

$$+\frac{e^2\cdot(g-2)^2}{4\cdot m^2}\cdot\frac{\gamma_P}{\gamma_P+1}\cdot(\vec{s}\wedge\vec{v}_P)^\dagger\cdot \vec{B}_P\cdot \vec{B}_P$$

$$-\frac{e^2\cdot g\cdot(g-2)}{4\cdot m^2}\cdot\frac{\gamma_P}{\gamma_P+1}\cdot \vec{v}_P^\dagger\cdot \vec{B}_P\cdot(\vec{s}\wedge\vec{v}_P)^\dagger\cdot \vec{B}_P\cdot \vec{v}_P$$

$$+\frac{e^2\cdot g^2}{4\cdot m^2}\cdot\frac{\gamma_P}{\gamma_P+1}\cdot \vec{v}_P^\dagger\cdot \vec{B}_P\cdot \vec{s}^\dagger\cdot \vec{v}_P\cdot(\vec{v}_P\wedge\vec{B}_P)\ . \tag{C.3}$$

From (2.3),(C.3) follows:

$$m\cdot(\gamma_M\cdot\vec{v}_M)' = e\cdot(\vec{v}_P\wedge\vec{B}_P)$$

$$+\frac{e\cdot c_2}{2\cdot m\cdot\gamma_P}\cdot\vec{\nabla}_P([\gamma_P\cdot\vec\sigma-\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]^\dagger\cdot\vec{B}_P)$$

$$+\frac{e}{2\cdot m}\cdot[\gamma_P\cdot\vec\sigma-\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]^\dagger\cdot\vec{B}_P'\cdot\vec{v}_P\cdot(\frac{2\cdot\gamma_P}{\gamma_P+1}+c_1\cdot\gamma_P)$$

$$+\frac{e}{2\cdot m}\cdot\vec{v}_P^\dagger\cdot\vec\sigma\cdot\vec{B}_P'\cdot\gamma_P\cdot(c_2-c_1-\frac{2}{\gamma_P+1}-g\cdot\frac{\gamma_P}{\gamma_P+1})$$

$$+\frac{e\cdot g}{2\cdot m}\cdot\frac{1}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P'\cdot[\gamma_P\cdot\vec\sigma-\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]$$

$$+\frac{e\cdot g}{2\cdot m}\cdot\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P'\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot\vec{v}_P$$

$$+\frac{e^2}{4\cdot m^2}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot(c_5-g^2\cdot\frac{1}{\gamma_P\cdot(\gamma_P+1)})\cdot\vec{B}_P\wedge[\gamma_P\cdot\vec\sigma-\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]$$

$$-\frac{e^2}{4\cdot m^2}\cdot c_5\cdot\vec{B}_P^\dagger\cdot\vec{B}_P\cdot\gamma_P\cdot(\vec{v}_P\wedge\vec\sigma)$$

$$+\frac{e^2}{4\cdot m^2}\cdot[\gamma_P\cdot\vec\sigma-\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot\vec{v}_P]^\dagger\cdot\vec{B}_P\cdot(\vec{v}_P\wedge\vec{B}_P)\cdot(2\cdot c_4-c_3-\frac{4\cdot\gamma_P}{\gamma_P+1})$$

$$+\frac{e^2\cdot(g-2)^2}{4\cdot m^2}\cdot\frac{\gamma_P^2}{\gamma_P+1}\cdot(\vec\sigma\wedge\vec{v}_P)^\dagger\cdot\vec{B}_P\cdot\vec{B}_P$$

$$-\frac{e^2\cdot g\cdot(g-2)}{4\cdot m^2}\cdot\frac{\gamma_P^2}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot(\vec\sigma\wedge\vec{v}_P)^\dagger\cdot\vec{B}_P\cdot\vec{v}_P$$

$$+\frac{e^2\cdot g^2}{4\cdot m^2}\cdot\frac{\gamma_P}{\gamma_P+1}\cdot\vec{v}_P^\dagger\cdot\vec{B}_P\cdot\vec\sigma^\dagger\cdot\vec{v}_P\cdot(\vec{v}_P\wedge\vec{B}_P)$$

$$=\ e\cdot(\vec{v}_P\wedge\vec{B}_P)+\frac{e\cdot c_2}{2\cdot m}\cdot\vec{\nabla}_M(\vec\sigma^\dagger\cdot\vec{B}_M)-\frac{e\cdot c_2}{2\cdot m}\cdot\frac{\gamma_M}{\gamma_M+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_M\cdot\vec{B}_M'$$

$$+\frac{e}{2\cdot m}\cdot[\gamma_M\cdot\vec\sigma-\frac{\gamma_M^2}{\gamma_M+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_M\cdot\vec{v}_M]^\dagger\cdot\vec{B}_M'\cdot\vec{v}_M\cdot(\frac{2\cdot\gamma_M}{\gamma_M+1}+c_1\cdot\gamma_M)$$

$$+\frac{e}{2\cdot m}\cdot\vec{v}_M^\dagger\cdot\vec\sigma\cdot\vec{B}_M'\cdot\gamma_M\cdot(c_2-c_1-\frac{2}{\gamma_M+1}-g\cdot\frac{\gamma_M}{\gamma_M+1})$$

$$+\frac{e\cdot g}{2\cdot m}\cdot\frac{1}{\gamma_M+1}\cdot\vec{v}_M^\dagger\cdot\vec{B}_M'\cdot[\gamma_M\cdot\vec\sigma-\frac{\gamma_M^2}{\gamma_M+1}\cdot\vec\sigma^\dagger\cdot\vec{v}_M\cdot\vec{v}_M]$$

$$+\frac{e \cdot g}{2 \cdot m} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (c_5 - g^2 \cdot \frac{1}{\gamma_M \cdot (\gamma_M + 1)}) \cdot \vec{B}_M \wedge [\gamma_M \cdot \vec{\sigma} - \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M]$$

$$-\frac{e^2}{4 \cdot m^2} \cdot c_5 \cdot \vec{B}_M^\dagger \cdot \vec{B}_M \cdot \gamma_M \cdot (\vec{v}_M \wedge \vec{\sigma})$$

$$+\frac{e^2}{4 \cdot m^2} \cdot [\gamma_M \cdot \vec{\sigma} - \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M]^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M) \cdot (2 \cdot c_4 - c_3 - \frac{4 \cdot \gamma_M}{\gamma_M + 1})$$

$$+\frac{e^2 \cdot (g-2)^2}{4 \cdot m^2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{B}_M$$

$$-\frac{e^2 \cdot g \cdot (g-2)}{4 \cdot m^2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{v}_M$$

$$+\frac{e^2 \cdot g^2}{4 \cdot m^2} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$= \quad e \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e \cdot c_2}{2 \cdot m} \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$+\frac{e}{2 \cdot m} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M' \cdot \gamma_M \cdot [\frac{\gamma_M}{\gamma_M + 1} \cdot (c_2 - g) - c_1 - \frac{2}{\gamma_M + 1}]$$

$$+\frac{e}{2 \cdot m} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M \cdot (\frac{2 \cdot \gamma_M^2}{\gamma_M + 1} + c_1 \cdot \gamma_M^2)$$

$$+\frac{e}{2 \cdot m} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot [\frac{\gamma_M}{\gamma_M + 1} \cdot (g-2) - c_1 \cdot \gamma_M]$$

$$+\frac{e \cdot g}{2 \cdot m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma}$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{B}_M \wedge \vec{\sigma}) \cdot \gamma_M \cdot (c_5 - g^2 \cdot \frac{1}{\gamma_M \cdot (\gamma_M + 1)})$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot (\vec{v}_M \wedge \vec{B}_M) \cdot [\frac{\gamma_M^2}{\gamma_M + 1} \cdot (c_3 - 2 \cdot c_4 + c_5) + g^2 \cdot \frac{\gamma_M^2}{(\gamma_M + 1)^2}$$

$$+4 \cdot \frac{\gamma_M^3}{(\gamma_M + 1)^2}] - \frac{e^2}{4 \cdot m^2} \cdot c_5 \cdot \vec{B}_M^\dagger \cdot \vec{B}_M \cdot \gamma_M \cdot (\vec{v}_M \wedge \vec{\sigma})$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M) \cdot \gamma_M \cdot (2 \cdot c_4 - c_3 - \frac{4 \cdot \gamma_M}{\gamma_M + 1})$$

$$+\frac{e^2 \cdot (g-2)^2}{4 \cdot m^2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{B}_M$$

$$-\frac{e^2 \cdot g \cdot (g-2)}{4 \cdot m^2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \ . \tag{C.4}$$

This can be simplified by calculating via (1.6-7),(B.4),(B.7),(B.12-13):

$$\vec{v}_P = \vec{v}_M + \vec{\Delta}_7 = \vec{v}_M + \frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M)' = \frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \left((\vec{\Omega}_M \wedge \vec{\sigma}) \wedge \vec{v}_M + \vec{\sigma} \wedge \vec{v}_M' \right)$$

$$= \quad \vec{v}_M + \frac{1}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \left(\vec{v}_M^\dagger \cdot \vec{\Omega}_M \cdot \vec{\sigma} - \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{\Omega}_M + \frac{e}{K_M} \cdot \vec{B}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M - \frac{e}{K_M} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{B}_M \right)$$

$$= \frac{e}{2 \cdot m^2} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \left(-\frac{g}{\gamma_M} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} + [\frac{2}{\gamma_M} + g - 2] \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{B}_M \right.$$

$$\left. -(g-2) \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + \frac{2}{\gamma_M} \cdot \vec{B}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M - \frac{2}{\gamma_M} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{B}_M \right)$$

$$= \vec{v}_M + \frac{e}{2 \cdot m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left(-g \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} + (g-2) \cdot \gamma_M \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{B}_M \right.$$

$$\left. -(g-2) \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + 2 \cdot \vec{B}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M \right), \tag{C.5}$$

from which follows by (B.21)

$$e \cdot (\vec{v}_P \wedge \vec{B}_P) = e \cdot (\vec{v}_M \wedge \vec{B}_P) + \frac{e^2}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left(-\frac{g}{2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \right.$$

$$\left. -\frac{g-2}{2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \right) \wedge \vec{B}_M$$

$$= e \cdot (\vec{v}_M \wedge \vec{B}_M) - e \cdot (\vec{v}_M \wedge \vec{\Delta}_{12}) + \frac{e^2}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left(-\frac{g}{2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \right.$$

$$\left. -\frac{g-2}{2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \right) \wedge \vec{B}_M . \tag{C.6}$$

This can be further simplified by calculating via (B.20),(B.30):

$$-e \cdot (\vec{v}_M \wedge \vec{\Delta}_{12}) = -e \cdot [\vec{v}_M \wedge \vec{\Delta}_{10}^\dagger \cdot \vec{\nabla}_M \vec{B}_M] = -e \cdot \vec{\Delta}_{10}^\dagger \cdot \vec{\nabla}_M (\vec{v}_M \wedge \vec{B}_M)$$

$$= -e \cdot \vec{\nabla}_M \left(\vec{\Delta}_{10}^\dagger \cdot (\vec{v}_M \wedge \vec{B}_M) \right) + e \cdot \vec{\Delta}_{10} \wedge \left(\vec{\nabla}_M \wedge (\vec{v}_M \wedge \vec{B}_M) \right)$$

$$= \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{\nabla}_M \left((\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot (\vec{v}_M \wedge \vec{B}_M) \right) + \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot (\vec{\sigma} \wedge \vec{v}_M) \wedge \vec{B}_M'$$

$$= \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M - \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{v}_M^\dagger \cdot \vec{v}_M)$$

$$+ \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot [\vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M - \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma}]$$

$$= \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot [-\vec{v}_M^\dagger \cdot \vec{v}_M \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M) + \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M' + \vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M - \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma}] ,$$

so that (C.6) reads as:

$$e \cdot (\vec{v}_P \wedge \vec{B}_P) = e \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot [\vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M' + \vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M$$

$$-\vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma} - \vec{v}_M^\dagger \cdot \vec{v}_M \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)] + \frac{e^2}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left(-\frac{g}{2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \right.$$

$$\left. -\frac{g-2}{2} \cdot \frac{\gamma_M^2}{\gamma_M + 1} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \right) \wedge \vec{B}_M .$$

Inserting this into (C.4) yields:

$$m \cdot (\gamma_M \cdot \vec{v}_M)' = e \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{m} \cdot \frac{\gamma_M}{\gamma_M + 1} \cdot [\vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M' + \vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M$$

$$-\vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma} - \vec{v}_M^\dagger \cdot \vec{v}_M \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)] + \frac{e^2}{m^2} \cdot \frac{1}{\gamma_M + 1} \cdot \left(-\frac{g}{2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma} \right.$$

$$-\frac{g-2}{2} \cdot \frac{\gamma_M^2}{\gamma_M+1} \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M + \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \Big) \wedge \vec{B}_M + \frac{e \cdot c_2}{2 \cdot m} \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$+\frac{e}{2 \cdot m} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M' \cdot \gamma_M \cdot [\frac{\gamma_M}{\gamma_M+1} \cdot (c_2 - g) - c_1 - \frac{2}{\gamma_M+1}]$$

$$+\frac{e}{2 \cdot m} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M \cdot (\frac{2 \cdot \gamma_M^2}{\gamma_M+1} + c_1 \cdot \gamma_M^2)$$

$$+\frac{e}{2 \cdot m} \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M \cdot \frac{\gamma_M^2}{\gamma_M+1} \cdot [\frac{\gamma_M}{\gamma_M+1} \cdot (g-2) - c_1 \cdot \gamma_M]$$

$$+\frac{e \cdot g}{2 \cdot m} \cdot \frac{\gamma_M}{\gamma_M+1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma}$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{B}_M \wedge \vec{\sigma}) \cdot \gamma_M \cdot (c_5 - g^2 \cdot \frac{1}{\gamma_M \cdot (\gamma_M+1)})$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot (\vec{v}_M \wedge \vec{B}_M) \cdot [\frac{\gamma_M^2}{\gamma_M+1} \cdot (c_3 - 2 \cdot c_4 + c_5) + g^2 \cdot \frac{\gamma_M^2}{(\gamma_M+1)^2}$$

$$+4 \cdot \frac{\gamma_M^3}{(\gamma_M+1)^2}] - \frac{e^2}{4 \cdot m^2} \cdot c_5 \cdot \vec{B}_M^\dagger \cdot \vec{B}_M \cdot \gamma_M \cdot (\vec{v}_M \wedge \vec{\sigma})$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M) \cdot \gamma_M \cdot (2 \cdot c_4 - c_3 - \frac{4 \cdot \gamma_M}{\gamma_M+1})$$

$$+\frac{e^2 \cdot (g-2)^2}{4 \cdot m^2} \cdot \frac{\gamma_M^2}{\gamma_M+1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{B}_M$$

$$-\frac{e^2 \cdot g \cdot (g-2)}{4 \cdot m^2} \cdot \frac{\gamma_M^2}{\gamma_M+1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{v}_M$$

$$= \ e \cdot (\vec{v}_M \wedge \vec{B}_M) + \frac{e}{2 \cdot m} \cdot (-\frac{2 \cdot \gamma_M}{\gamma_M+1} \cdot \vec{v}_M^\dagger \cdot \vec{v}_M + c_2) \cdot \vec{\nabla}_M (\vec{\sigma}^\dagger \cdot \vec{B}_M)$$

$$+\frac{e}{2 \cdot m} \cdot \Big([(c_2 - g) \cdot \frac{\gamma_M^2}{\gamma_M+1} - c_1 \cdot \gamma_M] \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{B}_M'$$

$$+(c_1 \cdot \gamma_M^2 + 2 \cdot \gamma_M) \cdot \vec{\sigma}^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M$$

$$+\frac{\gamma_M^2}{\gamma_M+1} \cdot [\frac{\gamma_M}{\gamma_M+1} \cdot (g-2) - c_1 \cdot \gamma_M] \cdot \vec{\sigma}^\dagger \cdot \vec{v}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{v}_M$$

$$+(g-2) \cdot \frac{\gamma_M}{\gamma_M+1} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M' \cdot \vec{\sigma} \Big)$$

$$+\frac{e^2}{4 \cdot m^2} \cdot \Big([4 - 4 \cdot \gamma_M + \gamma_M \cdot (2 \cdot c_4 - c_3)] \cdot \vec{B}_M^\dagger \cdot \vec{\sigma} \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$+\frac{\gamma_M^2}{\gamma_M+1} \cdot [-2 \cdot c_4 + c_3 + c_5 + 4 + (g^2 - 2 \cdot g) \cdot \frac{1}{\gamma_M+1}] \cdot \vec{v}_M^\dagger \cdot \vec{\sigma} \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{B}_M)$$

$$+[(g-2) \cdot g \cdot \frac{1}{\gamma_M+1} - \gamma_M \cdot c_5] \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot (\vec{\sigma} \wedge \vec{B}_M) - c_5 \cdot \gamma_M \cdot \vec{B}_M^\dagger \cdot \vec{B}_M \cdot (\vec{v}_M \wedge \vec{\sigma})$$

$$+(g-2)^2 \cdot \frac{\gamma_M^2}{\gamma_M+1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{B}_M$$

$$-(g^2 - 2 \cdot g) \cdot \frac{\gamma_M^2}{\gamma_M+1} \cdot (\vec{\sigma} \wedge \vec{v}_M)^\dagger \cdot \vec{B}_M \cdot \vec{v}_M^\dagger \cdot \vec{B}_M \cdot \vec{v}_M \Big) . \tag{C.7}$$

This completes the derivation of (8.15).

Appendix D

D.1

In this Appendix I show that (5.12) is equivalent to (5.5). First of all one observes by (3.1-2) that (5.12a) is equivalent to (5.5a).

D.2

In this subsection I show that (5.12b) is equivalent to (5.5b) and to do that I only have to show that the spatial part of (5.5b) is equivalent to (5.12b). [46] To come to that I have to calculate the spatial parts of several 4-vectors. Given arbitrary antisymmetric tensors $N, \hat{N}$ of rank 2 and a 4-vector a with the following notation:

$$N \leftrightarrow (\vec{b}, \vec{d}) , \qquad \hat{N} \leftrightarrow (\hat{\vec{b}}, \hat{\vec{d}}) , \qquad a_\mu = (\vec{a}^\dagger, a_4)_\mu , \qquad (\mu = 1, ..., 4)$$

one gets:

$$N_{\mu\nu} \cdot a_\nu = \left([\vec{a} \wedge \vec{b} + a_4 \cdot \vec{d}]^\dagger, -\vec{a}^\dagger \cdot \vec{d} \right)_\mu , \quad N_{\nu\omega} \cdot \hat{N}_{\omega\nu} = -2 \cdot \vec{b}^\dagger \cdot \hat{\vec{b}} - 2 \cdot \vec{d}^\dagger \cdot \hat{\vec{d}} . \qquad (\mu = 1, ..., 4)$$

$$\text{(D.1)}$$

One thus gets [47]

$$F^P_{\mu\nu} \cdot U^P_\nu = \gamma_P \cdot \left([\vec{v}_P \wedge \vec{B}_P + \vec{E}_P]^\dagger, i \cdot \vec{v}^\dagger_P \cdot \vec{E}_P \right)_\mu ,$$

$$F^P_{\mu\nu} \cdot F^P_{\nu\rho} \cdot U^P_\rho = \gamma_P \cdot \left([(\vec{v}_P \wedge \vec{B}_P + \vec{E}_P) \wedge \vec{B}_P + \vec{v}^\dagger_P \cdot \vec{E}_P \cdot \vec{E}_P]^\dagger, i \cdot (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P)^\dagger \cdot \vec{E}_P \right)_\mu ,$$

$$S^P_{\mu\nu} \cdot F^P_{\nu\rho} \cdot U^P_\rho$$
$$= \gamma_P \cdot \left([(\vec{v}_P \wedge \vec{B}_P + \vec{E}_P) \wedge \vec{s} - \vec{v}^\dagger_P \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})]^\dagger, -i \cdot (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P)^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \right)_\mu ,$$

$$U^P_\lambda \cdot \partial^P_\lambda = \gamma_P \cdot (\vec{v}^\dagger_P \cdot \vec{\nabla}_P + \partial/\partial t) , \qquad (\mu = 1, ..., 4)$$

and

$$U^P_\mu \to \gamma_P \cdot \vec{v}_P ,$$
$$\dot{U}^P_\mu \to \gamma_P \cdot (\gamma_P \cdot \vec{v}_P)' ,$$
$$F^P_{\mu\nu} \cdot U^P_\nu \to \gamma_P \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] ,$$
$$S^P_{\nu\omega} \cdot \partial^P_\mu F^P_{\omega\nu} \to -2 \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}^\dagger_P \cdot (\vec{s} \wedge \vec{v}_P) \right) ,$$
$$U^P_\mu \cdot S^P_{\nu\omega} \cdot F^P_{\omega\nu} \to -2 \cdot \gamma_P \cdot [\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}^\dagger_P \cdot (\vec{s} \wedge \vec{v}_P)] \cdot \vec{v}_P ,$$
$$U^P_\mu \cdot S^P_{\nu\omega} \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\omega\nu} \to -2 \cdot \gamma^2_P \cdot [\vec{s}^\dagger \cdot \vec{B}'_P - (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}'_P] \cdot \vec{v}_P ,$$
$$S^P_{\mu\nu} \cdot U^P_\omega \cdot U^P_\lambda \cdot \partial^P_\lambda F^P_{\nu\omega} \to \gamma^2_P \cdot [\vec{E}'_P + \vec{v}_P \wedge \vec{B}'_P - \vec{v}^\dagger_P \cdot \vec{E}'_P \cdot \vec{v}_P] \wedge \vec{s}$$

---

[46] The temporal part of equation (5.5b) follows from the spatial part of equation (5.5b) by using the constraints (3.8).

[47] The partial derivative $\partial/\partial t$ in this equation acts on functions depending on $\vec{r}_P, t$.

$$= \gamma_P^2 \cdot [\vec{E}'_P \wedge \vec{s} + \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}'_P - \vec{s}^\dagger \cdot \vec{B}'_P \cdot \vec{v}_P - \vec{v}_P^\dagger \cdot \vec{E}'_P \cdot (\vec{v}_P \wedge \vec{s})] \,,$$

$$S^P_{\mu\nu} \cdot F^P_{\nu\omega} \cdot F^P_{\omega\rho} \cdot U^P_\rho \to \gamma_P \cdot \left((\vec{v}_P \wedge \vec{B}_P + \vec{E}_P) \wedge \vec{B}_P + \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{E}_P \right) \wedge \vec{s}$$

$$-\gamma_P \cdot (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) = \gamma_P \cdot [\vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+\vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P - \vec{B}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P + \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s}) - (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})] \,,$$

$$F^P_{\mu\nu} \cdot S^P_{\nu\omega} \cdot F^P_{\omega\rho} \cdot U^P_\rho \to \gamma_P \cdot [(\vec{v}_P \wedge \vec{B}_P + \vec{E}_P) \wedge \vec{s} - \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})] \wedge \vec{B}_P$$

$$-\gamma_P \cdot (\vec{v}_P \wedge \vec{B}_P + \vec{E}_P)^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P = \gamma_P \cdot [-\vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) + \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$-(1 + \vec{v}_P^\dagger \cdot \vec{v}_P) \cdot \vec{B}_P^\dagger \cdot \vec{s} \cdot \vec{E}_P - \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} + \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{B}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P$$

$$+\vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P - \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P] \,,$$

$$F^P_{\mu\nu} \cdot U^P_\nu \cdot S^P_{\nu\omega} \cdot F^P_{\omega\nu} \to -2 \cdot \gamma_P \cdot [\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P)] \cdot [\vec{v}_P \wedge \vec{B}_P + \vec{E}_P] \,, \qquad \text{(D.2)}$$

where the expressions on the rhs of the arrows denote the corresponding spatial parts. With (D.2) the spatial part of the rhs of (5.5b) reads as:

$$\frac{e}{m} \cdot \gamma_P \cdot (\vec{v}_P \wedge \vec{B}_P) + \frac{e}{m} \cdot \gamma_P \cdot \vec{E}_P + \frac{e \cdot c_2}{2 \cdot m^2} \cdot \vec{\nabla}_P \left(\vec{s}^\dagger \cdot \vec{B}_P - \vec{E}_P^\dagger \cdot (\vec{s} \wedge \vec{v}_P) \right)$$

$$+\frac{e \cdot \gamma_P^2}{2 \cdot m^2} \cdot [(c_1 + 2) \cdot \vec{s}^\dagger \cdot \vec{B}'_P \cdot \vec{v}_P - c_2 \cdot (\vec{s} \wedge \vec{v}_P)^\dagger \cdot \vec{E}'_P \cdot \vec{v}_P + (c_2 - c_1 - 2) \cdot (\vec{E}'_P \wedge \vec{s})$$

$$+(c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{B}'_P - (c_2 - c_1 - 2) \cdot \vec{v}_P^\dagger \cdot \vec{E}'_P \cdot (\vec{v}_P \wedge \vec{s})]$$

$$+\frac{e^2}{4 \cdot m^3} \cdot \gamma_P \cdot [c_5 \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot (\vec{B}_P \wedge \vec{s}) - c_5 \cdot \vec{B}_P^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{s}) + c_5 \cdot \vec{E}_P^\dagger \cdot \vec{s} \cdot \vec{B}_P$$

$$+(-c_3 \cdot \vec{v}_P^\dagger \cdot \vec{v}_P - c_5 + 2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{E}_P + c_5 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot (\vec{E}_P \wedge \vec{s})$$

$$-c_5 \cdot (\vec{v}_P \wedge \vec{B}_P)^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s}) - c_5 \cdot \vec{E}_P^\dagger \cdot \vec{E}_P \cdot (\vec{v}_P \wedge \vec{s})$$

$$+(2 \cdot c_4 - c_3) \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot (\vec{v}_P \wedge \vec{B}_P) + c_3 \cdot \vec{E}_P^\dagger \cdot \vec{B}_P \cdot \vec{s} - c_3 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{s}$$

$$+c_3 \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}^\dagger \cdot \vec{B}_P \cdot \vec{v}_P + c_3 \cdot \vec{v}_P^\dagger \cdot \vec{s} \cdot \vec{v}_P^\dagger \cdot \vec{B}_P \cdot \vec{E}_P$$

$$+(2 \cdot c_4 - c_3) \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot \vec{E}_P + 2 \cdot c_4 \cdot \vec{E}_P^\dagger \cdot (\vec{v}_P \wedge \vec{s}) \cdot (\vec{v}_P \wedge \vec{B}_P)] \,, \qquad \text{(D.3)}$$

and the spatial part of the lhs of (5.5b) reads as:

$$\gamma_P \cdot (\gamma_P \cdot \vec{v}_P)' \,. \qquad \text{(D.4)}$$

By (D.3-4),(5.5b),(5.12b) one sees that multiplying the spatial part of (5.5b) by m/γ_P results in (5.12b) so that the spatial part of (5.5b) is equivalent to (5.12b). Thus I have shown that (5.5b) is equivalent to (5.12b).

D.3

In this subsection I show that the spatial part of (5.5c) is equivalent to (5.12c). To come to that I have to calculate the spatial parts of several antisymmetric tensors of rank 2. Using again the notation

$$a_\mu = (\vec{a}^\dagger, a_4)_\mu \,, \qquad (\mu = 1, ..., 4)$$

I calculate first of all: [48]

$$\frac{1}{2} \cdot \sum_{k,i=1}^{3} \varepsilon_{jki} \cdot [S^P_{k\omega} \cdot F^P_{\omega\lambda} \cdot a_\lambda \cdot a_i - S^P_{i\omega} \cdot F^P_{\omega\lambda} \cdot a_\lambda \cdot a_k] = \sum_{k,i=1}^{3} \varepsilon_{jki} \cdot S^P_{k\omega} \cdot F^P_{\omega\lambda} \cdot a_\lambda \cdot a_i$$

$$= \sum_{k,i=1}^{3} \varepsilon_{jki} \cdot [\sum_{m=1}^{3} S^P_{km} \cdot F^P_{m\lambda} \cdot a_\lambda \cdot a_i + S^P_{k4} \cdot F^P_{4\lambda} \cdot a_\lambda \cdot a_i]$$

$$= \sum_{k,i=1}^{3} \varepsilon_{jki} \cdot [\sum_{m,n=1}^{3} \varepsilon_{kmn} \cdot s_n \cdot F^P_{m\lambda} \cdot a_\lambda \cdot a_i + S^P_{k4} \cdot F^P_{4\lambda} \cdot a_\lambda \cdot a_i]$$

$$= \sum_{k,i=1}^{3} \varepsilon_{jki} \cdot \left(\sum_{m,n=1}^{3} \varepsilon_{kmn} \cdot s_n \cdot [\sum_{r=1}^{3} F^P_{mr} \cdot a_r \cdot a_i + F^P_{m4} \cdot a_4 \cdot a_i] + S^P_{k4} \cdot F^P_{4\lambda} \cdot a_\lambda \cdot a_i \right)$$

$$= \sum_{k,i=1}^{3} \varepsilon_{jik} \cdot \left(\sum_{m,n=1}^{3} \varepsilon_{nmk} \cdot s_n \cdot [\sum_{q,r=1}^{3} \varepsilon_{mrq} \cdot B_{P,q} \cdot a_r \cdot a_i + F^P_{m4} \cdot a_4 \cdot a_i] - \sum_{m=1}^{3} S^P_{k4} \cdot F^P_{4m} \cdot a_m \cdot a_i \right)$$

$$= \sum_{k,i=1}^{3} \varepsilon_{jik} \cdot \left(\sum_{m,n=1}^{3} \varepsilon_{nmk} \cdot s_n \cdot [\sum_{q,r=1}^{3} \varepsilon_{mrq} \cdot B_{P,q} \cdot a_r \cdot a_i - i \cdot E_{P,m} \cdot a_4 \cdot a_i] \right.$$

$$\left. - \sum_{m=1}^{3} q_k \cdot E_{P,m} \cdot a_m \cdot a_i \right)$$

$$= \sum_{i,m,n=1}^{3} [\delta_{jn} \cdot \delta_{im} - \delta_{jm} \cdot \delta_{in}] \cdot s_n \cdot [\sum_{q,r=1}^{3} \varepsilon_{mrq} \cdot B_{P,q} \cdot a_r \cdot a_i - i \cdot E_{P,m} \cdot a_4 \cdot a_i]$$

$$+ \vec{a}^\dagger \cdot \vec{E}_P \cdot (\vec{q} \wedge \vec{a})_j$$

$$= \sum_{i,q,r=1}^{3} \varepsilon_{irq} \cdot B_{P,q} \cdot a_r \cdot a_i \cdot s_j - \sum_{n,q,r=1}^{3} \varepsilon_{jrq} \cdot B_{P,q} \cdot a_r \cdot a_n \cdot s_n$$

$$- i \cdot a_4 \cdot \sum_{i=1}^{3} [E_{P,i} \cdot a_i \cdot s_j - E_{P,j} \cdot a_i \cdot s_i] + \vec{a}^\dagger \cdot \vec{E}_P \cdot (\vec{q} \wedge \vec{a})_j$$

$$= \vec{a}^\dagger \cdot \vec{s} \cdot (\vec{B}_P \wedge \vec{a})_j - i \cdot a_4 \cdot \vec{a}^\dagger \cdot \vec{E}_P \cdot s_j + i \cdot a_4 \cdot \vec{a}^\dagger \cdot \vec{s} \cdot E_{P,j} + \vec{a}^\dagger \cdot \vec{E}_P \cdot (\vec{q} \wedge \vec{a})_j \ ,$$

$$(j = 1, 2, 3) \qquad\qquad (D.5)$$

from which follows

$$S^P_{\mu\omega} \cdot F^P_{\omega\lambda} \cdot a_\lambda \cdot a_\nu - S^P_{\nu\omega} \cdot F^P_{\omega\lambda} \cdot a_\lambda \cdot a_\mu \rightarrow$$
$$\vec{a}^\dagger \cdot \vec{s} \cdot (\vec{B}_P \wedge \vec{a}) - i \cdot a_4 \cdot \vec{a}^\dagger \cdot \vec{E}_P \cdot \vec{s} + i \cdot a_4 \cdot \vec{a}^\dagger \cdot \vec{s} \cdot \vec{E}_P + \vec{a}^\dagger \cdot \vec{E}_P \cdot (\vec{q} \wedge \vec{a}) \ , \qquad (D.6)$$

where the expression on the rhs of the arrow denotes the corresponding spatial part. For the special case: $a = U^P$ one gets from (D.6):

$$S^P_{\mu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\nu - S^P_{\nu\omega} \cdot F^P_{\omega\lambda} \cdot U^P_\lambda \cdot U^P_\mu \rightarrow$$
$$-\gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot [\vec{E}_P + \vec{v}_P \wedge \vec{B}_P] + \vec{v}^\dagger_P \cdot \vec{E}_P \cdot \vec{s} + \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}^\dagger_P \cdot \vec{E}_P \cdot \vec{v}_P \ . \qquad (D.7)$$

Also I calculate:

$$\frac{1}{2} \cdot \sum_{k,l=1}^{3} \varepsilon_{jkl} \cdot [S^P_{k\omega} \cdot F^P_{\omega l} - S^P_{l\omega} \cdot F^P_{\omega k}] = \sum_{k,l=1}^{3} \varepsilon_{jkl} \cdot S^P_{k\omega} \cdot F^P_{\omega l}$$

[48]Note that: $\sum_{k=1}^{3} \varepsilon_{jpk} \cdot \varepsilon_{nmk} = \delta_{jn} \cdot \delta_{pm} - \delta_{jm} \cdot \delta_{pn}$ for $j, p, m, n = 1, 2, 3$.

$$= \sum_{k,l=1}^{3} \varepsilon_{jkl} \cdot [\sum_{m=1}^{3} S^P_{km} \cdot F^P_{ml} + S^P_{k4} \cdot F^P_{4l}] = \sum_{k,l=1}^{3} \varepsilon_{jkl} \cdot [\sum_{m,n=1}^{3} \varepsilon_{kmn} \cdot s_n \cdot F^P_{ml} + S^P_{k4} \cdot F^P_{4l}]$$

$$= \sum_{k,l=1}^{3} \varepsilon_{jlk} \cdot \left(\sum_{m,n,r=1}^{3} \varepsilon_{nmk} \cdot s_n \cdot \varepsilon_{mlr} \cdot B_r - S^P_{k4} \cdot F^P_{4l} \right)$$

$$= \sum_{k,l=1}^{3} \varepsilon_{jlk} \cdot \left(\sum_{m,n,r=1}^{3} \varepsilon_{nmk} \cdot s_n \cdot \varepsilon_{mlr} \cdot B_{P,r} - q_k \cdot E_{P,l} \right)$$

$$= \sum_{l,m,n,r=1}^{3} [\delta_{jn} \cdot \delta_{lm} - \delta_{jm} \cdot \delta_{ln}] \cdot s_n \cdot \varepsilon_{mlr} \cdot B_{P,r} + (\vec{q} \wedge \vec{E}_P)_j$$

$$= - \sum_{n,r=1}^{3} \varepsilon_{jnr} \cdot s_n \cdot B_{P,r} + (\vec{q} \wedge \vec{E}_P)_j = (\vec{B}_P \wedge \vec{s})_j + (\vec{q} \wedge \vec{E}_P)_j , \qquad (j = 1, 2, 3) \quad \text{(D.8)}$$

from which follows

$$F^P_{\mu\omega} \cdot S^P_{\omega\nu} - S^P_{\mu\omega} \cdot F^P_{\omega\nu} \rightarrow \vec{s} \wedge \vec{B}_P + (\vec{v}_P \wedge \vec{s}) \wedge \vec{E}_P , \tag{D.9}$$

where the expression on the rhs of the arrow denotes the corresponding spatial part. With (D.7),(D.9) the spatial part of the rhs of (5.5c) reads as:

$$\frac{e}{m} \cdot [\frac{g-2}{2} \cdot \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot (\vec{E}_P + \vec{v}_P \wedge \vec{B}_P) + \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{s}$$

$$-\frac{g-2}{2} \cdot \gamma_P^2 \cdot \vec{s}^\dagger \cdot \vec{v}_P \cdot \vec{v}_P^\dagger \cdot \vec{E}_P \cdot \vec{v}_P + \frac{g}{2} \cdot (\vec{s} \wedge \vec{B}_P) - \frac{g}{2} \cdot \vec{s}^\dagger \cdot \vec{E}_P \cdot \vec{v}_P] , \tag{D.10}$$

and the spatial part of the lhs of (5.5c) reads as:

$$\gamma_P \cdot \vec{s}\,' . \tag{D.11}$$

Thus I have shown that the spatial part of equation (5.5c) is equivalent to equation (5.12c). Using (3.6) one also finds that the temporal part of equation (5.5c) follows from equation (5.12c) so that equation (5.5c) is equivalent to (5.12c).

Appendix E

In this Appendix I show that (6.8) is equivalent to (5.5). From (6.2),(6.7) follows [49]

$$S^P_{\nu\alpha} \cdot F^P_{\alpha\rho} = \frac{1}{2} \cdot \varepsilon_{\nu\alpha\beta\gamma} \cdot \varepsilon_{\alpha\rho\lambda\sigma} \cdot T^P_\beta \cdot U^P_\gamma \cdot \tilde{F}^P_{\lambda\sigma} = -\frac{1}{2} \cdot \varepsilon_{\alpha\nu\beta\gamma} \cdot \varepsilon_{\alpha\rho\lambda\sigma} \cdot T^P_\beta \cdot U^P_\gamma \cdot \tilde{F}^P_{\lambda\sigma}$$

$$= -\frac{1}{2} \cdot [\delta_{\nu\rho} \cdot \delta_{\beta\lambda} \cdot \delta_{\gamma\sigma} + \delta_{\nu\sigma} \cdot \delta_{\beta\rho} \cdot \delta_{\gamma\lambda} + \delta_{\nu\lambda} \cdot \delta_{\beta\sigma} \cdot \delta_{\gamma\rho} - \delta_{\nu\rho} \cdot \delta_{\beta\sigma} \cdot \delta_{\gamma\lambda} - \delta_{\nu\sigma} \cdot \delta_{\beta\lambda} \cdot \delta_{\gamma\rho}$$

$$-\delta_{\nu\lambda} \cdot \delta_{\beta\rho} \cdot \delta_{\gamma\sigma}] \cdot T^P_\beta \cdot U^P_\gamma \cdot \tilde{F}^P_{\lambda\sigma} = \delta_{\nu\rho} \cdot U^P_\alpha \cdot T^P_\beta \cdot \tilde{F}^P_{\alpha\beta} - \tilde{F}^P_{\nu\alpha} \cdot T^P_\alpha \cdot U^P_\rho + \tilde{F}^P_{\nu\alpha} \cdot U^P_\alpha \cdot T^P_\rho ,$$

$$F^P_{\nu\alpha} \cdot S^P_{\alpha\rho} = \delta_{\nu\rho} \cdot U^P_\alpha \cdot T^P_\beta \cdot \tilde{F}^P_{\alpha\beta} - \tilde{F}^P_{\rho\alpha} \cdot T^P_\alpha \cdot U^P_\nu + \tilde{F}^P_{\rho\alpha} \cdot U^P_\alpha \cdot T^P_\nu ,$$

$$S^P_{\mu\nu} \cdot F^P_{\nu\omega} \cdot U^P_\omega = U^P_\mu \cdot U^P_\alpha \cdot T^P_\beta \cdot \tilde{F}^P_{\alpha\beta} + \tilde{F}^P_{\mu\nu} \cdot T^P_\nu ,$$

$$S^P_{\lambda\omega} \cdot F^P_{\omega\lambda} = 2 \cdot U^P_\alpha \cdot T^P_\beta \cdot \tilde{F}^P_{\alpha\beta} , \qquad (\mu, \nu, \rho = 1, ..., 4) \tag{E.1}$$

---

[49]Note that: $\varepsilon_{\alpha\nu\beta\gamma} \cdot \varepsilon_{\alpha\rho\lambda\sigma} = \delta_{\nu\rho} \cdot \delta_{\beta\lambda} \cdot \delta_{\gamma\sigma} + \delta_{\nu\sigma} \cdot \delta_{\beta\rho} \cdot \delta_{\gamma\lambda} + \delta_{\nu\lambda} \cdot \delta_{\beta\sigma} \cdot \delta_{\gamma\rho}$
$- \delta_{\nu\rho} \cdot \delta_{\beta\sigma} \cdot \delta_{\gamma\lambda} - \delta_{\nu\sigma} \cdot \delta_{\beta\lambda} \cdot \delta_{\gamma\rho} - \delta_{\nu\lambda} \cdot \delta_{\beta\rho} \cdot \delta_{\gamma\sigma}$ for $\nu, \beta, \gamma, \rho, \lambda, \sigma = 1, ..., 4$.

so that one gets by (5.5c),(6.1-2): [50]

$$\dot{T}_\mu^P = -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot \frac{d}{d\tau} \left(S_{\nu\rho}^P \cdot U_\omega^P \right) = -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot [\dot{S}_{\nu\rho}^P \cdot U_\omega^P + S_{\nu\rho}^P \cdot \dot{U}_\omega^P]$$

$$= -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot [\frac{e \cdot g}{2 \cdot m} \cdot (F_{\nu\alpha}^P \cdot S_{\alpha\rho}^P \cdot U_\omega^P - S_{\nu\alpha}^P \cdot F_{\alpha\rho}^P \cdot U_\omega^P) + \frac{e}{m} \cdot S_{\nu\rho}^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P]$$

$$= -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot [\frac{e \cdot g}{2 \cdot m} \cdot (-\tilde{F}_{\rho\alpha}^P \cdot T_\alpha^P \cdot U_\nu^P \cdot U_\omega^P + \tilde{F}_{\rho\alpha}^P \cdot U_\alpha^P \cdot T_\nu^P \cdot U_\omega^P + \tilde{F}_{\nu\alpha}^P \cdot T_\alpha^P \cdot U_\rho^P \cdot U_\omega^P$$

$$- \tilde{F}_{\nu\alpha}^P \cdot U_\alpha^P \cdot T_\rho^P \cdot U_\omega^P) - \frac{i \cdot e}{m} \cdot \varepsilon_{\nu\rho\beta\gamma} \cdot T_\beta^P \cdot U_\gamma^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P]$$

$$= -\frac{i}{2} \cdot \varepsilon_{\mu\nu\rho\omega} \cdot [-\frac{i \cdot e \cdot g}{4 \cdot m} \cdot (-\varepsilon_{\rho\alpha\beta\gamma} \cdot F_{\beta\gamma}^P \cdot T_\alpha^P \cdot U_\nu^P \cdot U_\omega^P + \varepsilon_{\rho\alpha\beta\gamma} \cdot F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\nu^P \cdot U_\omega^P$$

$$+ \varepsilon_{\nu\alpha\beta\gamma} \cdot F_{\beta\gamma}^P \cdot T_\alpha^P \cdot U_\rho^P \cdot U_\omega^P - \varepsilon_{\nu\alpha\beta\gamma} \cdot F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\rho^P \cdot U_\omega^P) - \frac{i \cdot e}{m} \cdot \varepsilon_{\nu\rho\beta\gamma} \cdot T_\beta^P \cdot U_\gamma^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P]$$

$$= \frac{e \cdot g}{8 \cdot m} \cdot \varepsilon_{\rho\nu\mu\omega} \cdot \varepsilon_{\rho\alpha\beta\gamma} \cdot [-F_{\beta\gamma}^P \cdot T_\alpha^P \cdot U_\nu^P \cdot U_\omega^P + F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\nu^P \cdot U_\omega^P]$$

$$- \frac{e \cdot g}{8 \cdot m} \cdot \varepsilon_{\nu\rho\mu\omega} \cdot \varepsilon_{\nu\alpha\beta\gamma} \cdot [F_{\beta\gamma}^P \cdot T_\alpha^P \cdot U_\rho^P \cdot U_\omega^P - F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\rho^P \cdot U_\omega^P]$$

$$- \frac{e}{2 \cdot m} \cdot \varepsilon_{\nu\rho\mu\omega} \cdot \varepsilon_{\nu\rho\beta\gamma} \cdot T_\beta^P \cdot U_\gamma^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P$$

$$= \frac{e \cdot g}{4 \cdot m} \cdot \varepsilon_{\rho\nu\mu\omega} \cdot \varepsilon_{\rho\alpha\beta\gamma} \cdot [-F_{\beta\gamma}^P \cdot T_\alpha^P \cdot U_\nu^P \cdot U_\omega^P + F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\nu^P \cdot U_\omega^P]$$

$$- \frac{e}{2 \cdot m} \cdot \varepsilon_{\nu\rho\mu\omega} \cdot \varepsilon_{\nu\rho\beta\gamma} \cdot T_\beta^P \cdot U_\gamma^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P$$

$$= \frac{e \cdot g}{4 \cdot m} \cdot \varepsilon_{\rho\nu\mu\omega} \cdot \varepsilon_{\rho\alpha\beta\gamma} \cdot F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\nu^P \cdot U_\omega^P - \frac{e}{2 \cdot m} \cdot \varepsilon_{\nu\rho\mu\omega} \cdot \varepsilon_{\nu\rho\beta\gamma} \cdot T_\beta^P \cdot U_\gamma^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P$$

$$= \frac{e \cdot g}{4 \cdot m} \cdot [(\delta_{\nu\alpha} \cdot \delta_{\mu\beta} \cdot \delta_{\omega\gamma} + \delta_{\nu\gamma} \cdot \delta_{\mu\alpha} \cdot \delta_{\omega\beta} + \delta_{\nu\beta} \cdot \delta_{\mu\gamma} \cdot \delta_{\omega\alpha} - \delta_{\nu\alpha} \cdot \delta_{\mu\gamma} \cdot \delta_{\omega\beta} - \delta_{\nu\gamma} \cdot \delta_{\mu\beta} \cdot \delta_{\omega\alpha}$$

$$- \delta_{\nu\beta} \cdot \delta_{\mu\alpha} \cdot \delta_{\omega\gamma}) \cdot F_{\beta\gamma}^P \cdot U_\alpha^P \cdot T_\nu^P \cdot U_\omega^P] - \frac{e}{m} \cdot [\delta_{\mu\beta} \cdot \delta_{\omega\gamma} - \delta_{\mu\gamma} \cdot \delta_{\omega\beta}] \cdot T_\beta^P \cdot U_\gamma^P \cdot F_{\omega\alpha}^P \cdot U_\alpha^P$$

$$= \frac{e \cdot g}{2 \cdot m} \cdot [F_{\alpha\beta}^P \cdot U_\alpha^P \cdot T_\beta^P \cdot U_\mu^P + F_{\mu\alpha}^P \cdot T_\alpha^P] + \frac{e}{m} \cdot F_{\beta\alpha}^P \cdot T_\beta^P \cdot U_\alpha^P \cdot U_\mu^P$$

$$= \frac{e \cdot g}{2 \cdot m} \cdot F_{\mu\nu}^P \cdot T_\nu^P + \frac{e \cdot (g-2)}{2 \cdot m} \cdot U_\mu^P \cdot T_\omega^P \cdot F_{\nu\omega}^P \cdot U_\nu^P . \qquad (\mu = 1, ..., 4) \qquad \text{(E.2)}$$

Also one gets from (E.1):

$$S_{\nu\omega}^P \cdot \partial_\mu^P F_{\omega\nu}^P + U_\mu^P \cdot S_{\nu\omega}^P \cdot U_\lambda^P \cdot \partial_\lambda^P F_{\omega\nu}^P = 2 \cdot U_\alpha^P \cdot T_\beta^P \cdot \partial_\mu^P \tilde{F}_{\alpha\beta}^P + 2 \cdot U_\mu^P \cdot U_\alpha^P \cdot T_\beta^P \cdot U_\lambda^P \cdot \partial_\lambda^P \tilde{F}_{\alpha\beta}^P ,$$

$$S_{\mu\nu}^P \cdot U_\omega^P \cdot U_\lambda^P \cdot \partial_\lambda^P F_{\nu\omega}^P = U_\mu^P \cdot U_\alpha^P \cdot T_\beta^P \cdot U_\lambda^P \cdot \partial_\lambda^P \tilde{F}_{\alpha\beta}^P + T_\nu^P \cdot U_\lambda^P \cdot \partial_\lambda^P \tilde{F}_{\mu\nu}^P ,$$

$$F_{\mu\nu}^P \cdot S_{\nu\omega}^P \cdot F_{\omega\rho}^P \cdot U_\rho^P = F_{\mu\nu}^P \cdot U_\nu^P \cdot U_\alpha^P \cdot T_\beta^P \cdot \tilde{F}_{\alpha\beta}^P + F_{\mu\nu}^P \cdot \tilde{F}_{\nu\alpha}^P \cdot T_\alpha^P ,$$

$$F_{\mu\nu}^P \cdot U_\nu^P \cdot S_{\omega\rho}^P \cdot F_{\rho\omega}^P = 2 \cdot F_{\mu\nu}^P \cdot U_\nu^P \cdot U_\alpha^P \cdot T_\beta^P \cdot \tilde{F}_{\alpha\beta}^P ,$$

$$S_{\mu\nu}^P \cdot F_{\nu\rho}^P \cdot F_{\rho\omega}^P \cdot U_\omega^P = F_{\mu\nu}^P \cdot U_\nu^P \cdot U_\alpha^P \cdot T_\beta^P \cdot \tilde{F}_{\alpha\beta}^P - \tilde{F}_{\mu\nu}^P \cdot U_\nu^P \cdot U_\alpha^P \cdot T_\beta^P \cdot F_{\alpha\beta}^P . \qquad (\mu = 1, ..., 4)$$

$$\text{(E.3)}$$

Combining (5.5),(6.8),(E.2-3) one observes that (6.8) is equivalent to (5.5).

---

[50]Note that: $\varepsilon_{\nu\rho\mu\omega} \cdot \varepsilon_{\nu\rho\beta\gamma} = \delta_{\mu\beta} \cdot \delta_{\omega\gamma} - \delta_{\mu\gamma} \cdot \delta_{\omega\beta}$ for $\mu, \omega, \beta, \gamma = 1, ..., 4$.

Remarks accompanying the text

[a]: For a particle with 'normal' intrinsic magnetic dipole moment one has by definition: $g = 2$.

[b]: Transformations have been given in [Bl62, DS70] and in [CaM55] for special cases and Jackson mentions the 'Pauli reduction' [Jac76]. The general transformation is straightforward and has been carried out by the author. It will be published separately [Hei]. Note that it is consistent with the (semi-relativistic) Foldy-Wouthuysen transformation [BD64, CoM95, FW50, Fol62], because expanding the Hamiltonian $H_{M,op}$ up to second order in $1/c$ yields by (0.1) for $g = 2$:

$$H_{M,op} = \frac{m}{2} \cdot \beta + \frac{1}{4 \cdot m} \cdot \beta \cdot \vec{\pi}^{\dagger}_{M,op} \cdot \vec{\pi}_{M,op} - \frac{1}{16 \cdot m^3} \cdot \beta \cdot (\vec{\pi}^{\dagger}_{M,op} \cdot \vec{\pi}_{M,op})^2 + \frac{e}{2} \cdot \phi_{M,op}$$
$$- \frac{e}{2 \cdot m} \cdot \beta \cdot \vec{\sigma}^{\dagger}_{op} \cdot \vec{B}_{M,op} + \frac{e}{4 \cdot m^2} \cdot \vec{\sigma}^{\dagger}_{op} \cdot (\vec{\pi}_{M,op} \wedge \vec{E}_{M,op}) + \text{hermitian conjugate} .$$

The Darwin term does not appear here because of the *vacuum* Maxwell equations (see section 2). Expanding $H_{M,op}$ only up to first order in $1/c$ yields for $g = 2$:

$$H_{M,op} = m \cdot \beta + \frac{1}{2 \cdot m} \cdot \beta \cdot \vec{\pi}^{\dagger}_{M,op} \cdot \vec{\pi}_{M,op} + e \cdot \phi_{M,op} - \frac{e}{m} \cdot \beta \cdot \vec{\sigma}^{\dagger}_{op} \cdot \vec{B}_{M,op} ,$$

which has the form of the Schroedinger-Pauli Hamiltonian.

[c]: Thus the DK equations turn out to be Poincaré covariant but not *manifestly* Poincaré covariant whereas the Frenkel equations are manifestly covariant.

[d]: The particle described in this work has arbitrary but nonvanishing charge, arbitrary intrinsic magnetic dipole moment and vanishing intrinsic electric dipole moment. For a neutral particle the equations can be easily modified.

[e]: Here ε_{ijk} is the antisymmetric symbol with $\varepsilon_{123} = 1$ and δ denotes the Kronecker delta. All three-component quantities $\vec{r}_M, \vec{p}_M, \vec{\sigma}, \ldots$ denoted by an arrow are *column* vectors. The components a_j of any $\vec{a}$ are defined by:

$$\vec{a} = (a_1, a_2, a_3)^{\dagger} .$$

The transpose of a three-component quantity is denoted by '†'. Therefore $\vec{r}^{\dagger}_M, \vec{p}^{\dagger}_M, \vec{\sigma}^{\dagger}, \ldots$ are *row* vectors.

[f]: In this paper the nabla operator $\vec{\nabla}_M$ always acts on functions depending on $\vec{r}_M, t, \vec{p}_M, \vec{\sigma}$ and it is the gradient w.r.t. $\vec{r}_M$.

[g]: Note that in this paper the multiplication symbol '·' always denotes *matrix* multiplication and that a single number is a 1×1 matrix.

To avoid the mushrooming of the bracket symbol I avoid its use even in places where it would usually help to find the correct order of matrix multiplications to be performed. If for example a matrix product like

$$\vec{\pi}^{\dagger}_M \cdot \vec{B}_M \cdot \vec{\pi}_M$$

occurs (in this example associativity of multiplication does not hold!), then the matrix structures of the factors suggest the correct order of the matrix multiplications. In the present example one has:

$$\vec{\pi}_M^\dagger \cdot \vec{B}_M \cdot \vec{\pi}_M = (\vec{\pi}_M^\dagger \cdot \vec{B}_M) \cdot \vec{\pi}_M$$

because

$$\vec{\pi}_M^\dagger \cdot (\vec{B}_M \cdot \vec{\pi}_M)$$

is a meaningless expression. Note also that the multiplication symbol '∧' always the denotes the vector product of three-component quantities.

[h]: The nabla operator $\vec{\nabla}_P$ in this paper always acts on functions depending on $\vec{r}_P, t, \vec{v}_P, \vec{s}$ and it is the gradient w.r.t. $\vec{r}_P$. My notation is chosen so as to indicate that the functional dependence of $\vec{E}_P$ on $\vec{r}_P, t$ is the same as the dependence of $\vec{E}_M$ on $\vec{r}_M, t$ and likewise for $\vec{B}_M$. The explicit way in which the 'P' fields are derived from the 'M' fields is shown in (B.21).

[i]: I am dealing with special relativistic space-time positions, tensors, pseudotensors, tensor fields and pseudotensor fields and I do so by using the *complex* notation where the fourth (=temporal) component is imaginary. I use this convention following the usage in much of the literature on the relativistic SG force [Cor68, Fre26, Goo62, Nyb62, Nyb64, Pla66a, Pla66b, Raf70]. Note also that Einstein's summation conventions are applied to Greek indices. The Greek indices assume the values 1,2,3,4. One of the advantages of the complex convention is that only lower components occur. For textbooks using this convention, see for example [Moe72, Syn58]. Of course one could use the *real* convention where, however, one has to distinguish between covariant and contravariant components. Most textbooks use the real convention, e.g. [Jac75].

[j]: Any antisymmetric tensor N of rank 2 can be characterized by two three-component quantities $\vec{b}$ resp. $\vec{d}$, called the spatial resp. temporal part of the tensor, and they are defined by:

$$b_j = \frac{1}{2} \cdot \sum_{k,l=1}^3 \varepsilon_{jkl} \cdot N_{kl} , \qquad d_j = N_{j4} . \qquad (j = 1, 2, 3)$$

I denote this correspondence as follows:

$$N \leftrightarrow (\vec{b}, \vec{d}) .$$

Thus the spatial resp. temporal part of M is given by $\vec{\mu}_P$ resp. $i \cdot \vec{\varepsilon}_P$, i.e.

$$M \leftrightarrow (\vec{\mu}_P, i \cdot \vec{\varepsilon}_P) .$$

[k]: The constraint (3.8b) follows from (3.2),(3.6) and states that the particle has no intrinsic electric dipole moment [Cor68, Fre26, Nyb64].

[l]: In this paper the partial derivatives $\partial_1^P, ..., \partial_4^P$ always act on functions depending on $X_1^P, X_2^P, X_3^P, X_4^P$.

[m]: One also gets this Hamiltonian if one neglects terms of second order in spin in the Hamiltonian given by Corben. For details, see [Cor68], especially Chapter 7 thereof. If one neglects all spin terms then H^P reduces to the following well known expression:

$$H^P = \frac{1}{2 \cdot m} \cdot \Pi_\mu^P \cdot \Pi_\mu^P + \frac{m}{2} \ .$$

See for example [Ba64, Gol80, Jac75].

[n]: The assumptions made on Y^P are as follows. The proper time dependence in the SG term on the rhs of (5.1b) is assumed to come in only via the proper time dependence of X^P, U^P, S^P. To get a useful class of allowed Y^P, I assume in addition that the X^P dependence only comes in via F^P and its space-time derivatives. Specifically I assume that $Y_1^P, ..., Y_4^P$ are functions of the following arguments: $m, U_1^P, ..., U_4^P$, the six components of S^P and all space time derivatives of the six components of F^P. This function is supposed to be a polynomial in all its arguments (except m) and first order in spin. Thus the coefficients of this polynomial are functions of m and it turns out that they are just powers of m times dimensionless numbers.

[o]: Another remark on the quantum mechanical aspect is in order. In deriving the classical equations from the Dirac equation (plus the Pauli term) one does not get a unique answer because one depends on the choice of the proper operators. For example the Frenkel equations can be obtained by using a certain special relativistic generalization of the Foldy-Wouthuysen transformation [Bl62, DS70, DS72, Hei] with the emphasis on the Newton-Wigner position operator whereas the GNR equations can be derived by the 'Gordon decomposition' [Raf70]. Note again that in the present work I am only interested in the classical aspect. Further details on the quantum mechanics including the opinions of Pauli and Bohr are discussed for example in [DSV86, Goo62, Roh72].

[p]: The Lorentz group consists of the homogeneous part of the Poincaré group, i.e. it does not contain the space-time translations. Note that the Poincaré group is also called the inhomogeneous Lorentz group. A Poincaré transformation is composed of a translation a and a Lorentz transformation L so that the space-time position X^P transforms as:

$$X_\mu^P \to L_{\mu\nu} \cdot X_\nu^P + a_\mu \ . \qquad (\mu = 1, ..., 4)$$

Since L belongs to the Lorentz group it satisfies

$$L_{\mu\nu} \cdot L_{\mu\rho} = \delta_{\nu\rho} \ . \qquad (\nu, \rho = 1, ..., 4)$$

The restricted Poincaré group (=proper orthochronous Poincaré group) contains those Poincaré transformations, where:

$$\det(L) = 1 \ , \qquad L_{44} > 0 \ .$$

Hence the restricted Poincaré group contains neither the parity transformation nor the time reversal transformation. For more details on the subgroups of the Lorentz group resp. Poincaré group, see for example [BLT75, SW89].

[q]: This follows because the relations (3.1),(3.6),(3.9) between the variables X^P, S^P, F^P and the variables $\vec{r}_P, t, \vec{s}, \vec{B}_P, \vec{E}_P$ are the same for every Poincaré frame.

[r]: If one performs a Poincaré transformation, then (as shown in subsection 5.3) the equations of motion (2.11) for the 'P' variables remain the same (except that the 'P' fields have transformed in a specified way). Under the same Poincaré transformation the equations of motion (1.5) for the 'M' variables also transform in a definite way. The transformed equations of motion for the 'M' variables can be derived from the transformed equations of motion for the 'P' variables in the same way as the original equations of motion (1.5) for the 'M' variables were derived in section 2 from the original equations of motion (2.11) for the 'P' variables because the relations (2.1),(2.3a),(2.4-5),(2.7) between the 'M' variables and the 'P' variables are the same for every Poincaré frame. The result is that the transformed equations of motion for the M' variables are the same as the original equations of motion (1.5) for the 'M' variables (except that the 'M' fields have transformed in a specified way). Therefore the DK equations (1.5) are Poincaré covariant.

Acknowledgements

References

[BHR94a] D.P. Barber, K. Heinemann, G. Ripken, Z. Phys., **C64**, p.117 (1994).

[BHR94b] D.P. Barber, K. Heinemann, G. Ripken, Z. Phys., **C64**, p.143 (1994).

[BHR] D.P. Barber, K. Heinemann, G. Ripken. A further work on SG forces in storage rings in preparation.

[BMT59] V. Bargmann, L. Michel, V.L. Telegdi, Phys. Rev. Lett., **2**, p.435 (1959).

[Ba64] A.O. Barut, "Electrodynamics and classical theory of fields and particles", New York (1964).

[BD64] J.D. Bjorken, S.D. Drell, "Relativistic quantum mechanics", New York (1964).

[Bl62] E.I. Blount, Phys. Rev., **126**, p.1636, **128**, p.2454 (1962).

[BLT75] N.N. Bogolubov, A.A. Logunov, I.T. Todorov, "Introduction to axiomatic quantum field theory", Reading (1975).

[BT80] V.A. Bordovitsyn, I.M. Ternov, Sov. Phys. Usp., **23**, p.679 (1980).

[Br95] R. Brinkmann, "HERA STATUS AND PLANS", presented at 16th IEEE Particle Accelerator Conference (PAC 95), Dallas, Texas, 1-5 May 1995.

[CaM55] K.M. Case, H. Mendlowitz, Phys. Rev., **97**, p.33 (1955).

[CPP95] M. Conte, A. Penzo, M. Pusterla, Nuovo Cim., **108A**, p.127 (1995).

[CJKP96] M. Conte, R. Jagannathan, S.A. Khan, M. Pusterla, Part. Acc., **56**, p.99 (1996).

[Cor68] H.C.. Corben, "Classical and quantum theories of spinning particles", San Francisco (1968).

[Cos94] J.P. Costella, Ph.D. thesis, The University of Melbourne (1994).

[CoM94] J.P. Costella, B.H.J. McKellar, Int. J. Mod. Phys., **A9**, p.461 (1994).

[CoM95] J.P. Costella, B.H.J. McKellar, Am. J. Phys., **63**, p.1119 (1995).

[DS70] S.R. de Groot, L.D. Suttorp, Nuovo Cim., **65A**, p.245 (1970).

[DS72] S.R. de Groot, L.D. Suttorp, "Foundations of electrodynamics", Amsterdam (1972).

[DSV86] R.S. Van Dyck, P.B. Schwinberg, H.G. Dehmelt, Phys. Rev., **D34**, p.722 (1986).

[DK73] Ya.S. Derbenev, A.M. Kondratenko, Sov. Phys. JETP, **37**, p.968 (1973).

[Der90a] Ya.S. Derbenev, University of Michigan - Ann Arbor, preprint, UM-HE **90-30** (1990).

[Der90b] Ya.S. Derbenev, University of Michigan - Ann Arbor, preprint, UM-HE **90-32** (1990).

[Der95] Ya.S. Derbenev, "Concepts for Stern-Gerlach Polarization in Storage Rings", seminar at DESY (May 1995).

[DH95] This was also conjectured by G.H. Hoffstätter and Ya.S. Derbenev.

[FW50] L. Foldy, S.A. Wouthuysen, Phys. Rev., **78**, p.29 (1950).

[Fol62] L. Foldy, in: "Quantum theory, Vol. 3: Radiation and high energy physics." ed. by D.R. Bates, New York (1962).

[FG61a] D.M. Fradkin, R.H. Good, Nuovo Cim., **22**, p.643 (1961).

[FG61b] D.M. Fradkin, R.H. Good, Rev. Mod. Phys., **33**, p.343 (1961).

[Fre26] J. Frenkel, Z. Phys., **37**, p.243 (1926).

[Gol80] H. Goldstein, "Classical mechanics", Reading (1980).

[Goo62] R.H. Good, Phys. Rev., **125**, p.2112 (1962).

[Hei] K. Heinemann, to be published (about the FW transformation).

[HW63] J. Hilgevoord, S.A. Wouthuysen, Nucl. Phys., **40**, p.1 (1963).

[Hof95] I wish to thank G.H. Hoffstätter for pointing this out to me in 1995.

[Hof96] G.H. Hoffstätter, private note: "Synchrotron motion with longitudinal Stern-Gerlach kicks" (1996).

[Jac75] J.D. Jackson, "Classical electrodynamics", New York (1975).

[Jac76] J.D. Jackson, Rev. Mod. Phys., **48**, p.417 (1976).

[JM63] T.F. Jordan, N. Mukunda, Phys. Rev., **132**, p.1842 (1963).

[Moe49] C. Møller, Ann. Inst. H. Poincaré, **11**, p.251 (1949).

[Moe72] C. Møller, "The theory of relativity", Oxford (1972).

[NW49] T.A. Newton, E.P. Wigner, Rev. Mod. Phys., **21**, p.400 (1949).

[NR87] T.O. Niinikoski, R. Rossmanith, Nucl. Inst. Meth., **A255**, p.460 (1987).

[Nyb62] P. Nyborg, Nuovo Cim., **23**, p.47 (1962).

[Nyb64] P. Nyborg, Nuovo Cim., **31**, p.1209 (1964), **32**, p.1131 (1964).

[Pla66a] E. Plahte, Supp. Nuovo Cim., **4**, p.246 (1966).

[Pla66b] E. Plahte, Supp. Nuovo Cim., **4**, p.291 (1966).

[Pry49] M.H.L. Pryce, Proc. Roy. Soc. London **A195**, p.62 (1949).

[Raf70] K. Rafanelli, Nuovo Cim., **67A**, p.48 (1970).

[Roh72] F. Rohrlich, "The Electron: Development of the First Elementary Particle Theory", Proceedings of a Symposium held at Trieste 1972, p.331.

[Sch30] E. Schroedinger, Berl. Ber., **1930**, p.418 (1930).

[SW89] R.F. Streater, A.S. Wightman, "PCT, spin and statistics, and all that", Redwood City (1989).

[ST64] A.A. Sokolov, I.M. Ternov, Sov. Phys. Doklady, **8**, p.1203 (1964).

[Syn58] J.L. Synge, "Relativity: The special theory", Amsterdam (1958).

[TVW80] C. Teitelboim, D. Villarroel, C. van Weert, Riv. Nuovo Cim., **3.9**, p.1 (1980).

[Tho27] L.H. Thomas, Phil. Mag., **3**, p.1 (1927).

[Yok87] K. Yokoya, Nucl. Instr. Meth., **A258**, p.149 (1987).